Parallel domain decomposition procedures of improved D–D type for parabolic problems

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A R T I C L E   I N F O

Article history:
Received 19 November 2008
Received in revised form 8 November 2009

MSC:
65N12
65N15
65N30
65N55

Keywords:
Parallel algorithm
Domain decomposition
Parabolic problem
A priori error estimate

A B S T R A C T

Two parallel domain decomposition procedures for solving initial-boundary value problems of parabolic partial differential equations are proposed. One is the extended D–D type algorithm, which extends the explicit/implicit conservative Galerkin domain decomposition procedures, given in [5], from a rectangle domain and its decomposition that consisted of a stripe of sub-rectangles into a general domain and its general decomposition with a net-like structure. An almost optimal error estimate, without the factor $H^{-1/2}$ given in Dawson–Dupont’s error estimate, is proved. Another is the parallel domain decomposition algorithm of improved D–D type, in which an additional term is introduced to produce an approximation of an optimal error accuracy in $L^2$-norm.

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1. Introduction

Domain decomposition and parallel computation have become powerful tools for solving a large scale PDE systems. Domain decomposition methods decompose a large or complicated domain into several small or regular sub-domains with overlapping or non-overlapping, on which subproblems may be solved by the use of effective or fast algorithms. Recently, a lot of algorithms based on overlapping and non-overlapping domain decompositions have been proposed to solve parabolic problems, for example, see [1–22,26,27]. In [5], Dawson and Dupont proposed some explicit/implicit conservative Galerkin domain decomposition procedures to solve parabolic problems and also proved the a priori error estimates. We call these procedures D–D type algorithms. Their numerical experiments showed an almost optimal error accuracy in $L^2$-norm. Since the earliest work in [5], D–D type algorithms have been developed and applied to many applications in Engineering. But, in the a priori error estimate in [5], there was a loss of $H^{-1/2}$, i.e., error estimates were not optimal in $L^2$-norm. The authors said that they did not know how to improve it. On the other hand, D–D type algorithms only were designed for a problem defined a rectangle domain and its domain decomposition consisted of a stripe of sub-rectangles, in which the inner boundaries of sub-domains are parallel to the same coordinate. So these procedures are not suitable to problems defined on a domain of a general shape and of a general domain decomposition without a chain structure.

The purpose of this article is to propose parallel domain decomposition algorithms on a general domain and its general decompositions. One is the extended D–D type algorithm. Particularly, the D–D procedure is a special case of the extended D–D type algorithm. Our analysis shows that an approximation given by the extended D–D type algorithm has an almost optimal error accuracy in $L^2$-norm. As a consequence, we prove a new error estimate of Dawson–Dupont’s domain.
decomposition procedure, in which there is no a loss of $H^{-1/2}$ order. Numerical results in [5] have confirmed these theoretical analyses. Furthermore, in order to obtain an approximation of an optimal error accuracy, we propose the improved D–D type domain decomposition algorithm.

The organization of the article is as follows. In Section 2, we propose two D–D type domain decomposition procedures. One is the extended D–D type algorithm, which is designed for a problem defined on a general domain and its decomposition without a chain structure assumption. Another is the improved D–D type algorithm, in which an additional term is introduced to improve approximating accuracy of numerical solutions. Section 3 is devoted to an analysis of $L^2$-norm error estimate for the extended and improved D–D type domain decomposition procedures described in Section 2. We prove that the extended D–D type domain decomposition algorithm reaches to accuracy of an almost optimal order in $L^2$-norm. Furthermore, we prove that the improved D–D type domain decomposition algorithm has an optimal error estimate in $L^2$-norm.

2. D–D type domain decomposition algorithms

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, with a boundary $\partial \Omega$. We consider the initial-boundary value problem of parabolic partial differential equation of the form:

\[
\begin{align*}
(a) & \quad \frac{\partial u}{\partial t} - \nabla \cdot (a \nabla u) + bu = f, & \text{in } \Omega \times (0, T]; \\
(b) & \quad \frac{\partial u}{\partial n} = 0, & \text{on } \partial \Omega \times (0, T]; \\
(c) & \quad u = u_0, & \text{in } \Omega, \text{ at } t = 0;
\end{align*}
\]

where $n$ is the unit vector outward normal to $\partial \Omega$, coefficients $a = a(x) \in C^1(\Omega)$, $b = b(x)$, $f = f(x, t)$ and $u_0 = u_0(x)$ are some given real-valued functions.

We consider a domain decomposition of many sub-domains. Divide $\Omega$ into $I_j$ non-overlapping sub-domains, $\{\Omega_j\}_{j=1}^J$, such that $\Omega = \bigcup_{j=1}^J \Omega_j$, and $\Gamma = \bigcup_{j=1}^J \Gamma_j$, which is the set of the inner boundaries of sub-domains. Here we do not require the domain and its decomposition to be rectangles and to be of a chain structure. In order to apply D–D type algorithms, we assume that

\begin{enumerate}[label=(H\arabic*)]
\item[(H1)] Inner-boundaries $\Gamma_j$ of $\Omega_j, 1 \leq j \leq J$, are piecewise linear.
\item[(H2)] There exists a constant $H_0 > 0$ such that $\Omega_j = \{y : y = x - \tau n_{\Gamma_j}, x \in \Gamma_j, 0 \leq \tau \leq H_0\} \subset \Omega_j$.
\end{enumerate}

where $n_{\Gamma_j}$ is the unit vector outward normal to $\Gamma_j$ for $1 \leq j \leq J$.

Under Condition (H1), it is convenient to construct finite element spaces on sub-domains. As well Condition (H2) is not a severe constraint, which only requires that inner boundaries of sub-domains have included angles larger than or equal to $\pi/2$ at their intersection points.

For $j = 1, 2, \ldots, J$, let $T_j^h$ be a family of triangulation of $\Omega_j$ and $M_j^h$ be a family of piecewise linear finite element spaces defined on $T_j^h$ having the approximate property that,

\[
\inf_{v_h \in M_j} \left\{ \|u - u_h\|_{L^2(\Omega_j)} + h \|\nabla (u - u_h)\|_{L^2(\Omega_j)} \right\} \leq Ch^2 \|u\|_{H^2(\Omega_j)}. \tag{2.2}
\]

Set $T^h = \bigcup_{j=1}^J T_j^h$. Let $M_j^h$ be the subspace of $L^2(\Omega_j)$ such that $v_h \in M_j^h$, if and only if $v_h |_{\Omega_j} \in M_j^h$ for each $1 \leq j \leq J$. For any point at $\Gamma_j$, assign $n_{\Gamma_j}$ as the given unit vector normal to $\Gamma_j$. Along the direction $n_{\Gamma_j}$, a function $v$ in $M_j^h$ has a well-defined jump $[v]$ across $\Gamma_j$:

\[ [v](x) = k_i \lim_{s \to 0^+} v(x - sn_{\Gamma_j}) + k_j \lim_{s \to 0^+} v(x - sn_{\Gamma_j}), \quad \forall x \in \Gamma_j \bigcap \Gamma_j, \]
where \( \lim_{\tau \to 0^+} v(x - sn_{r_j}) \) is the inner limit of the function along the outward normal direction \( n_{r_j} \), and

\[
\kappa_i = (n_{r_j} \cdot n_{r_i}) = \begin{cases} 1, & \text{as } n_{r_j} = n_{r_i}; \\ -1, & \text{as } n_{r_j} = -n_{r_i}. \end{cases}
\]

For each \( 0 < H \leq H_0 \), define the function as

\[
\varphi(\tau) = \begin{cases} 1 + \tau, & \text{for } -1 \leq \tau \leq 0; \\ 1 - \tau, & \text{for } 0 \leq \tau \leq 1; \\ 0, & \text{for } |\tau| > 1 \end{cases}
\]

and

\[
\phi(\tau) = \frac{1}{H} \varphi\left(\frac{\tau}{H}\right), \quad \tau \in \mathbb{R}^1.
\]

Integrating by part,

\[
\int_{-H}^{H} \phi(\tau) \nabla u(x + \tau n_{r_i}) \cdot n_{r_i} d\tau = \int_{-H}^{H} \phi(\tau) u'(x + \tau n_{r_i}) d\tau = -\int_{-H}^{H} \phi'(\tau) u(x + \tau n_{r_i}) d\tau,
\]

we define an approximate of a normal direction derivative as follows:

\[
B(\psi)(x) = -\int_{-H}^{H} \phi'(\tau) \psi(x + \tau n_{r_i}) d\tau, \quad x \in \Omega_j \cap \Omega_i, \quad 1 \leq i < j \leq J.
\]

It is clear that \( B(\psi) \) is the weighted averaging of the normal direction derivative and with the weighted function \( \phi \), which is an approximation of \( \delta \)-function. The condition (H2) insures that \( B(\psi) \) is of the certain definition. Noting that

\[
- \sum_{j=1}^{J} (\nabla \cdot (a \nabla u), v)_{\Omega_j} = \sum_{j=1}^{J} (a \nabla u, \nabla v)_{\Omega_j} - \sum_{j=1}^{J} (a \nabla u \cdot n_{r_j}, v)_{\Gamma_j}
\]

and

\[
\sum_{j=1}^{J} (a \nabla u \cdot n_{r_j}, v)_{\Gamma_j} = \sum_{i<j} (a \nabla u \cdot n_{r_i}, [v])_{\Gamma_i \cap \Gamma_j} = (a \nabla u \cdot n_{r_i}, [v])_{\Gamma_i}
\]

and defining the bilinear form \( D(\cdot, \cdot) \):

\[
D(\psi, \rho) = \sum_{j=1}^{J} \int_{\Omega_j} a \nabla \psi \cdot \nabla \rho \, dx,
\]

we have

\[
\left( \partial_t u, v_h \right) + D(u, v_h) + (bu, v_h) - (a \nabla u \cdot n_{r_i}, [v_h])_{\Gamma_i} = (f, v_h), \quad \forall \, v_h \in M^h.
\]

From (2.7), we derive the extended D–D type domain decomposition algorithm in Section 2.1 and the improved D–D type domain decomposition algorithm in Section 2.2, respectively.

2.1. Extended D–D type domain decomposition algorithm

Let \( 0 = t_0 < t_1 < \cdots < t_N = T \) be a time partition, \( \Delta t^n = t^n - t^{n-1}, \Delta t = \max_{1 \leq n \leq N} \Delta t^n, u^n(x, y) = u(x, y, t^n) \) and \( \tilde{\partial}_t U^n = (U^n - U^{n-1})/\Delta t^n \). Replacing with (2.5) the direction derivative on the inner boundary \( \Gamma' \), the extended D–D type domain decomposition algorithm reads:

**Extended domain decomposition procedure (EDDP)**

Give an initial approximation, \( U^0 \in M^h \), of \( u_0 \).

Seek \( U^n \in M^h \) such that

\[
\left( \tilde{\partial}_t U^n, V \right) + D(U^n, V) + (bu^n, V) - (a B(U^{n-1}), [V])_{\Gamma} = (f^n, V), \quad \forall \, V \in M^h
\]

for \( n = 1, 2, \ldots, N \).

Once \( U^{n-1} \) is given, the procedure (2.8) could be performed in sub-domains \( \{\Omega_j\}_{j=1}^J \) in parallel.

**Equivalent parallel algorithm of EDDP**

Give an initial approximation, \( U^0 \in M^h \), of \( u_0 \).

Seek \( U^n \in M^h \), for \( n = 1, \ldots, N \), in two steps:
Step 1. Seek $U^n_j \in \mathcal{M}_j^h$, for $1 \leq j \leq J$, in parallel, such that

\[
(U^n_j, V)_{\Omega_j} + \Delta t^n D(U^n_j, V)_{\Omega_j} + \Delta t^n (bU^n_j, V)_{\Omega_j} = (U^{n-1} + \Delta t^n f^n, V)_{\Omega_j} + \Delta t^n (aB(U^{n-1}), \kappa_j V)_{\Gamma_j}, \quad \forall V \in \mathcal{M}_j^h. \tag{2.9}
\]

Step 2. Define

\[
U^n(x) = U^n_j(x), \quad x \in \Omega_j, \quad 1 \leq j \leq J.
\tag{2.10}
\]

To obtain error estimate of the algorithm (EDDP), we introduce an elliptic projection, $W \in \mathcal{M}_h$, of the solution $u$ as follows:

\[
D(u(\cdot, t) - W(\cdot, t), V) + \kappa (u(\cdot, t) - W(\cdot, t), V) = 0, \quad \forall V \in \mathcal{M}_h, \quad 0 \leq t \leq T,
\tag{2.11}
\]

where $\kappa \geq 63 \|\sqrt{\kappa}\|_{W^{1,\infty}} + 1$. The parameter $\kappa$ will be used in the analysis of the convergence. Denote the error function of the projection by

\[
\eta = u - W.
\tag{2.12}
\]

For the algorithm (EDDP), we have the following convergence theorem.

**Theorem 2.1.** Assume that the solution $u$ is sufficiently smooth. Let $\{U^n\}_{n=1}^N$ be the solution of the algorithm (EDDP) and $U^0 \in \mathcal{M}_h$ be the interpolation or $L^2$-projection of $u_0$. The following a priori error estimate

\[
\max_{1 \leq n \leq N} \|u^n - U^n\|_{L^2(\Omega)} \leq C \left\{ \|\eta\|_{L^2([0,T], L^\infty(\Omega))} + \|\eta_t\|_{L^2([0,T], L^\infty(\Omega))} + \Delta t + \frac{h^2}{4a} \right\}
\tag{2.13}
\]

holds, provided

\[
\Delta t \leq \frac{h^2}{4a}.
\tag{2.14}
\]

The proof of Theorem 2.1 is put into Section 3.2. Here we will prove that (2.13) is almost optimal. It is clear that the auxiliary problem (2.11) is equivalent to

\[
D(W, V)_{\Omega_j} + \kappa (W, V)_{\Omega_j} = \left( f - \frac{\partial u}{\partial t} + (\kappa - b) u, V \right)_{\Omega_j} + \left( a \frac{\partial u}{\partial n_{\Gamma_j}}, V \right)_{\Gamma_j}, \quad \forall V \in \mathcal{M}_j^h, \quad 1 \leq j \leq J.
\tag{2.15}
\]

These are some standard finite element equations. Theorem 2.1 shows that the $L^2$-norm error estimate reaches the same order accuracy as the $L^\infty$-norm error of the standard finite element method for elliptic problems. In the case of the linear finite element space, from [23,24,17,18], we see the following $L^\infty$-error estimate:

\[
\|\eta\|_{L^\infty(\Omega)} + \|\eta_t\|_{L^\infty(\Omega)} \leq C h^2 \ln h \left\{ \|u\|_{W^2,\infty(\Omega)} + \|u_t\|_{W^2,\infty(\Omega)} \right\}.
\tag{2.16}
\]

As a consequence, we conclude that the error estimate (2.13) is almost optimal.

**Remark.** It is clear that the algorithm given in [5] is the particular case of the algorithm (EDDP), in which $a \equiv 1$, $\Omega = [0, 1] \times [0, 1]$ and $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1 = [0, \frac{1}{2}] \times [0, 1]$ and $\Omega_2 = [\frac{1}{2}, 1] \times [0, 1]$. Dawson and Dupont proved the following error estimate:

\[
\max_{1 \leq n \leq N} \|u^n - U^n\|_{L^2(\Omega)} \leq C \left\{ \Delta t + H^2 + \int_0^T \|\eta_t\|_{L^\infty(\Omega_1)} dt + \int_0^T \|\eta_t\|_{L^\infty(\Omega \setminus \Omega_1)} dt \right\}.
\tag{2.17}
\]

In the error estimate (2.17), there is a loss of accuracy of $H^{-1/2}$ order. But their numerical experiments showed an accuracy of higher order than this estimate. Theorem 2.1 shows there is no a loss of $H^{-1/2}$-order, which were supported by numerical experiments in [5].

### 2.2. Improved D–D type domain decomposition algorithm

Theorem 2.1 shows that the approximate accuracy of the solution of (EDDP) is almost optimal in $L^2$-norm, which depends upon $L^\infty$-norm error estimate. To obtain an algorithm of an optimal order accuracy, we propose the improved D–D type domain decomposition procedure, which produces an approximation with an optimal $L^2$-norm error accuracy independent of $L^\infty$-norm error estimates so that it could be applied to other problems without good $L^\infty$-norm error estimates.

**Improved Domain Decomposition Procedure (IDDPP)**

*Give an initial approximation, $U^0 \in \mathcal{M}_h$, of $u_0$.***
Seek \( U^n \in \mathcal{M}^h \) such that
\[
(\bar{\partial} U^n, V) + D(U^n, V) + (b U^n, V) - (a B(U^{n-1}), [V])_{\Gamma} - (a B(V), [U^{n-1}])_{\Gamma} = (f^n, V), \quad \forall V \in \mathcal{M}^h
\]  \hspace{1cm} \text{(2.18)}
for \( n = 1, 2, \ldots, N \).

In (IDDP), only an additional term, \(- (a B(V), [U^{n-1}])_{\Gamma}\), is introduced. Once \( U^{n-1} \) is given, this procedure could also be performed in sub-domains \( \{\Omega_j\}_{j=1}^5 \) in parallel. For the algorithm (IDDP), we have the following convergence theorem.

**Theorem 2.2.** Assume that the solution \( u \) is sufficiently smooth. Let \( \{U^n\}_{n=1}^N \) be the solution of the algorithm (IDDP) and that \( U^0 \in \mathcal{M}^h \) be the interpolation or \( L^2 \)-projection of \( u_0 \). Then there exists a constant \( C \) independent of \( h, \Delta t \) and \( H \) such that
\[
\max_{1 \leq n \leq N} \|u^n - U^n\|_{L^2(\Omega)} \leq C \left\{ \Delta t + h^2 + H^2 \right\}, \quad \text{(2.19)}
\]
provided
\[
\Delta t \leq \frac{H^2}{4a_1} \left( 1 - \sqrt{\frac{2}{3}} \right). \quad \text{(2.20)}
\]

The proof of Theorem 2.2 is put into Section 3.3. Theorem 2.2 shows the optimal error accuracy in \( L^2 \)-norm without the factor \(|\ln h|\).

### 3. Convergence analysis

In this section, we prove convergence theorems given in Section 2. We first give some lemmas in the Section 3.1, which will be used to prove the convergence results. Then Theorems 2.1 and 2.2 are proved in Sections 3.2 and 3.3, respectively.

#### 3.1. Some lemmas

Similar results have been given in [5], but here we give some finer conclusions. First of all, we have some results about \( B(\psi) \).

**Lemma 3.1.** For each \( \psi \in \mathcal{M}^h \), there holds the following inequality
\[
(a B(\psi), B(\psi))_{\Gamma} \leq 2^{3-2/p} a_1 H^{-2-2/p} |\Gamma|^{1-2/p} \|\psi\|_{H^1(\Omega)}^2, \quad 2 \leq p \leq \infty,
\]  \hspace{1cm} \text{(3.1)}
where \(|\Gamma|\) is the length of \( \Gamma \).

**Proof.** It follows from the definition of \( B(\psi) \) that for each \( x \in \Gamma_i \cap \Gamma_j, (1 \leq i < j) \)
\[
|B(\psi)(x)|^2 = \left| \int_{-H}^{H} \phi'(\tau) \psi(x + \tau n_f) d\tau \right|^2 \leq 2^{2-2/p} H^{-2-2/p} \left( \int_{-H}^{H} |\psi(x + \tau n_f)|^p d\tau \right)^{2/p}, \quad \forall 1 \leq p \leq \infty,
\]
which leads to (3.1). \( \blacksquare \)

**Lemma 3.2.** For each \( \psi \in \mathcal{M}^h \), there holds the equality that for any \( x \in \Gamma \)
\[
B(\psi)(x) = -\frac{1}{H} [\psi](x) + \int_{-H}^{H} \phi(\tau) \nabla \psi(x + \tau n_f) \cdot n_f d\tau + \int_{-H}^{0} \phi(\tau) \nabla \psi(x + \tau n_f) \cdot n_f d\tau. \quad \text{(3.2)}
\]

**Proof.** By using \( \phi(-H) = \phi(H) = 0, \int_{-H}^{H} \phi d\tau = 1 \) and integrating by part, we have
\[
- \int_{-H}^{H} \phi'(\tau) \psi(x + \tau n_f) d\tau = \frac{1}{H} \left( \lim_{\tau \to 0^+} \psi(x + \tau n_f) - \lim_{\tau \to 0^-} \psi(x + \tau n_f) \right)
+ \int_{0}^{H} \phi(\tau) \nabla \psi(x + \tau n_f) \cdot n_f d\tau + \int_{-H}^{0} \phi(\tau) \nabla \psi(x + \tau n_f) \cdot n_f d\tau.
\]
Noting that
\[
\lim_{\tau \to 0^+} \psi(x + \tau n_f) - \lim_{\tau \to 0^-} \psi(x + \tau n_f) = - \left\{ (n_f, n_{f_j}) \lim_{\tau \to 0^+} \psi(x - \tau n_{f_j}) + (n_f, n_{f_i}) \lim_{\tau \to 0^+} \psi(x - \tau n_{f_i}) \right\}
= -[\psi](x), \quad \forall x \in \Gamma_i \cap \Gamma_j, \quad 1 \leq i, j \leq J.
\]
we derive (3.2). \( \blacksquare \)
As consequences of Lemma 3.2, we have the following results

**Lemma 3.3.** For each \( \psi \in \mathcal{M}^h \), there holds the following inequality:

\[
\| \mathcal{B}(\psi) \|_{L^2(\Omega)} \leq CH^{-\frac{1}{2}} \left( H^{-1}\|\psi\|_{L^2(\Gamma)}^2 + \sum_{j=1}^n \| \nabla \psi \|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \quad \forall \psi \in \mathcal{M}^h
\]

(3.3)

and

\[
\| \| \psi \|_{L^2(\Gamma)} \leq H^{-1/2}\|\psi\|_{L^2(\Gamma)} + H^{1-p/2}\|\nabla \psi\|_{L^2(\Gamma)}, \quad \forall 2 \leq p \leq \infty.
\]

(3.4)

**Proof.** The inequality (3.3) follows directly from Lemma 3.2. It follows from Lemma 3.2 again that

\[
\frac{1}{H}[\psi](x) = -\mathcal{B}(\psi)(x) + \int_{-H}^H \phi(\tau)\psi'(x + \tau n)\,d\tau
\]

such that

\[
\| \| \psi \|_{L^2(\Gamma)} \leq H \left( \| \mathcal{B}(\psi) \|_{L^2(\Gamma)} + \int_{-H}^H \phi(\tau)\psi'(x + \tau n)\,d\tau \right).
\]

Noting (3.1) and

\[
\int_\Gamma \int_{-H}^H \phi(\tau)\psi'(x + \tau n)\,d\tau \,ds \leq \int_\Gamma \left( \int_{-H}^H \phi(\tau)\psi'(x + \tau n)\,d\tau \right)^{2/p} \left( \int_{-H}^H |\psi'(x + \tau n)|^p\,d\tau \right)^{2/p} \,ds \leq CH^{-2/p} \int_\Gamma \left( \int_{-H}^H |\psi'(x + \tau n)|^p\,d\tau \right)^{2/p} \,ds \leq CH^{-2/p}\|\nabla \psi\|_{L^2(\Omega)}^2.
\]

we have

\[
\| \| \psi \|_{L^2(\Gamma)} \leq H^{-1/2}\|\psi\|_{L^2(\Gamma)} + H^{1-p/2}\|\nabla \psi\|_{L^2(\Gamma)}.
\]

This ends the proof of Lemma 3.3. \( \blacksquare \)

**Lemma 3.4.** Let \( G \) be a domain containing the inner boundary \( \Gamma \) such that \( \text{supp}(\phi) \subset G \) for small \( 0 < H \leq H_0 \). If \( \psi \) is smooth in \( G \), we have estimates:

\[
\| \mathcal{B}(\psi) - \nabla \psi \cdot n \|_{L^2(\Gamma)} \leq C H \| \psi \|_{W^{2,\infty}(G)}
\]

(3.5)

and

\[
\| \mathcal{B}(\psi) - \nabla \psi \cdot n \|_{L^2(\Gamma)} \leq C H^2 \| \psi \|_{W^{3,\infty}(G)}.
\]

(3.6)

**Proof.** Let \( g(\tau) = \psi(x + \tau n) \). By integration by part, we have

\[
-\int_{-H}^H \phi(\tau)g(\tau)\,d\tau = -\phi(H)g(H) + \int_{-H}^H \phi(\tau)g'(\tau)\,d\tau
\]

\[
= g'(0) + \int_{-H}^H \phi(\tau)(g'(\tau) - g'(0))\,dx = g'(0) + \int_{-H}^H \phi(\tau) \left( \int_0^\tau g''(s)\,ds \right)\,d\tau,
\]

where we have used \( \phi(-H) = \phi(H) = 0, \int_{-H}^H \phi(\tau)\,d\tau = 1 \). This leads to

\[
B(\psi)(x) = \nabla \psi(x) \cdot n + \int_{-H}^H \phi(\tau) \int_0^\tau \frac{\partial^2 \psi}{\partial n^2}(x + \tau n)\,ds\,d\tau.
\]

(3.7)

So (3.5) holds.

Furthermore, we have

\[
\int_{-H}^H \phi(\tau) \int_0^\tau g''(s)\,ds\,d\tau = g''(0) \int_{-H}^H \phi(\tau)\,d\tau + \int_{-H}^H \phi(\tau) \int_0^\tau (g''(s) - g''(0))\,ds\,d\tau
\]

\[
= \int_{-H}^H \phi(\tau) \int_0^\tau (g''(s) - g''(0))\,ds\,d\tau = \int_{-H}^H \phi(\tau) \int_0^\tau (\tau - s)g''(s)\,ds\,d\tau.
\]
where we used \( \int_{-H}^H \phi(\tau)p(\tau)d\tau = p(0) \) for each polynomial of degree at most one such that \( \int_{-H}^H \phi(\tau)d\tau = 0 \). This means that
\[
B(\psi)(x) = \nabla \psi(x) \cdot n_F + \int_{-H}^H \phi(\tau) \int_0^\tau (\tau - s) \frac{\partial^3 \psi}{\partial n^3_F}(x + s n_F)dsd\tau.
\] (3.8)

(3.6) is derived. The proof of Lemma 3.4 is complete. ■

**Lemma 3.5.** For each \( \beta > 0 \), the following inequality
\[
-(aB(\psi), [\psi])_F \geq \left( 1 - \frac{11}{24} \right) \frac{1}{H} (a[\psi], [\psi])_F - \frac{5}{6\beta} \sum_{j=1}^l (a\nabla\psi, \nabla\psi)_{\partial j} - C_1 \|\psi\|_{L^2(\Omega)}^2
\] holds, where \( C_1 = 63 \|\sqrt{a}\|_{W^{1,\infty}(\Omega)}^2 \beta^{-1} \).

**Proof.** It follows from Lemma 3.2 that
\[
-\sqrt{a(x)}B(\psi)(x) = \frac{1}{H} \sqrt{a(x)}[\psi](x) - \int_{-H}^H \phi(\tau) \sqrt{a(x + \tau n_F)} \psi'(x + \tau n_F)d\tau
\]
\[
- \int_{-H}^H \phi(\tau) \left( \sqrt{a(x + \tau n_F)} - \sqrt{a(x)} \right) + \phi(\tau) \left( \sqrt{a(x + \tau n_F)} - \sqrt{a(x)} \right)' \psi(x + \tau n_F)d\tau
\]
\[
\geq \frac{1}{H}[\psi](x) - \sqrt{\frac{2}{3H}} \left( \int_{-H}^H a(x + \tau n_F) |\nabla\psi(x + \tau n_F)|^2d\tau \right)^{1/2}
\]
\[
- \sqrt{\frac{2}{H}} \|\sqrt{a}\|_{W^{1,\infty}(\Omega)} \left( 1 + \sqrt{\frac{1}{3}} \right) \left( \int_{-H}^H \psi^2(x + \tau n_F)d\tau \right)^{1/2}.
\]
Noting the fact that integrations near the intersections of inner boundaries may be calculated repeatedly two times at most under Condition (H2), we derive
\[
-(aB(\psi), [\psi])_F \geq \left( 1 - \frac{\beta}{\sqrt{6}} (1 + \delta) \right) \frac{1}{H} (a[\psi], [\psi])_F - \frac{2}{\sqrt{6\beta}} \sum_{j=1}^l (a\nabla\psi, \nabla\psi)_{\partial j}
\]
\[
- \sqrt{\frac{6}{\delta\beta}} \|\sqrt{a}\|_{W^{1,\infty}(\Omega)}^2 \left( 1 + \frac{1}{\sqrt{3}} \right) \|\psi\|_{L^2(\Omega)}^2.
\] (3.10)

Noting \( 2.4 \leq \sqrt{6} \leq 2.5 \) and taking \( \delta = 0.1 \), we derive
\[
-(aB(\psi), a[\psi])_F \geq \left( 1 - \frac{11}{24} \right) \frac{1}{H} (a[\psi], [\psi])_F - \frac{5}{6\beta} \sum_{j=1}^l (a\nabla\psi, \nabla\psi)_{\partial j} - C_1 \|\psi\|_{L^2(\Omega)}^2.
\]

We obtain (3.9). The proof of Lemma 3.5 is complete. ■

In next subsections, we prove Theorems 2.1 and 2.2, respectively.

### 3.2. Proof of Theorem 2.1

In this subsection, we prove Theorem 2.1. To this end, we introduce another elliptic projection, \( \tilde{W} \in \mathcal{M}^h \), of the solution \( u \) as follows:
\[
D(u - \tilde{W}, v) + \kappa (u - \tilde{W}, v) - (aB(u - \tilde{W}), [v])_F = 0, \quad \forall v \in \mathcal{M}^h, \quad 0 < t \leq T.
\] (3.11)

For a function \( \psi \) with restrictions in \( \bigcup_{j=1}^l H^1(\Omega_j) \), define a norm
\[
\|\psi\|_F^2 = (a\nabla\psi, \nabla\psi) + (\psi, \psi) + H^{-1}(a[\psi], [\psi])_F.
\] (3.12)

By taking \( \beta = 1 \) in Lemma 3.5, we see
\[
\frac{1}{6} \|v\|^2 \leq D(v, v) + \kappa (v, v) - (aB(v), [v])_F, \quad \forall v \in \mathcal{M}^h.
\]
It follows from Lax–Milgram theorem that the project problem (3.11) has a unique solution. Let
\[
\xi^n = u^n - \tilde{W}^n, \quad \theta^n = U^n - \tilde{W}^n.
\] (3.13)
We estimate \(\xi^n\) and \(\theta^n\) respectively in next lemmas.

**Lemma 3.6.** \(\{\theta^n\}_{n=0}^{\infty}\) satisfies the following estimate
\[
\max_{1 \leq n \leq N} \left\| \theta^n \right\|_{L^2(\Omega)}^2 \leq C \left\{ \Delta t^n \left( \left\| \partial_t \xi^n \right\|_{L^2(\Omega)}^2 + \left\| \xi^n \right\|_{L^2(\Omega)}^2 \right) + \Delta t^2 + H^5 \right\},
\] (3.14)
provided
\[
\Delta t \leq \frac{H^2}{4a_1}.
\] (3.15)

**Proof.** First of all, we see that \(u\) satisfies
\[
(\partial_t u^n, V) + D(u^n, V) + (bu^n, V) - \left( a \frac{\partial u^n}{\partial n}, [V] \right)_\Gamma = (f^n + \rho^n, V), \quad \forall \ V \in \mathcal{M}^h,
\] (3.16)
where the time truncation term \(\rho^n\) satisfies
\[
\sum_{n=1}^{M} \|\rho^n\|_{L^2(\Omega)}^2 \Delta t^n \leq \Delta t^2 \int_0^T \|u_t\|_{L^2(\Omega)}^2 dt.
\] (3.17)

It follows from (3.16) and the definition of \(\tilde{W}\) that
\[
(\partial_t \tilde{W}^n, V) + D(\tilde{W}^n, V) + (b\tilde{W}^n, V) - (aB(\tilde{W}^n), [V])_\Gamma = (f^n + (\kappa + b)\xi^n + \rho^n - \tilde{\alpha}_t \xi^n, V) - \left( a \left( B(u^n) - \frac{\partial u^n}{\partial n} \right), [V] \right)_\Gamma, \quad \forall \ V \in \mathcal{M}^h.
\] (3.18)
(2.8) and (3.18) give
\[
(\partial_t \theta^n, V) + D(\theta^n, V) - (aB(\theta^n-1), [V])_\Gamma = (\partial_t \xi^n - \rho^n - (\kappa + b)\xi^n - b\theta^n, V) - \left( a \left( B(u^n) - \frac{\partial u^n}{\partial n} \right), [V] \right)_\Gamma
\] + \(aB(\xi^n - \xi^{n-1}), [V])_\Gamma - (aB(u^n - u^{n-1}), [V])_\Gamma, \quad \forall \ V \in \mathcal{M}^h.
\] (3.19)
Taking \(V = \theta^n\) in (3.19), we have
\[
(\partial_t \theta^n, \theta^n) + D(\theta^n, \theta^n) - (aB(\theta^n), [\theta^n])_\Gamma = -(aB(\theta^n - \theta^{n-1}), [\theta^n])_\Gamma + (\partial_t \xi^n - \rho^n - (\kappa + b)\xi^n - b\theta^n, \theta^n)
\] - \(a \left( \frac{\partial u^n}{\partial n} - B(u^n) \right), [\theta^n] \right)_\Gamma + (aB(\xi^n - \xi^{n-1}), [\theta^n])_\Gamma - (aB(u^n - u^{n-1}), [\theta^n])_\Gamma.
\] (3.20)
Noting that
\[
(\partial_t \theta^n, \theta^n) = \frac{1}{2\Delta t^n} \left( \|\theta^n\|_{L^2(\Omega)}^2 - \|\theta^{n-1}\|_{L^2(\Omega)}^2 \right)
\] and substituting this equality into (3.20) and then summing it over \(n\), we get
\[
\|\theta^n\|_{L^2(\Omega)}^2 + \sum_{k=1}^n \left( \|\theta^k - \theta^{k-1}\|_{L^2(\Omega)}^2 \right) + 2\Delta t^n \left( D(\theta^k, \theta^k) - (aB(\theta^k), [\theta^k])_\Gamma \right)
\] = \(\|\theta^0\|_{L^2(\Omega)}^2 + 2\sum_{k=1}^n \Delta t^k \left( -(aB(\theta^k - \theta^{k-1}), [\theta^k])_\Gamma + (\partial_t \xi^k - \rho^k - (\kappa + b)\xi^k - b\theta^k, \theta^k)
\] - \(a \left( \frac{\partial u^k}{\partial n} - B(u^k) \right), [\theta^k] \right)_\Gamma + (aB(\xi^k - \xi^{k-1}), [\theta^k])_\Gamma - (aB(u^k - u^{k-1}), [\theta^k])_\Gamma.
\] (3.21)
For the third term on the left-hand side of (3.21), from Lemma 3.5, we see
\[
2(D(\theta^k, \theta^k) - (aB(\theta^k), [\theta^k])_\gamma) \geq \frac{1}{3} \sum_{j=1}^f (a \nabla \theta^k, \nabla \theta^k)_{\gamma} + \frac{13}{12H} (a[\theta^k], [\theta^k])_\gamma - C \| \theta^k \|^2_{L^2(\Omega)}.
\]
We estimate the terms on the right-hand side of (3.21) one by one. It follows from Lemma 3.4 that
\[
2 \left| \sum_{k=1}^n \Delta t^k (aB(\theta^k - \theta^{k-1}), [\theta^k])_\gamma \right| \leq 2 \Delta t \sum_{k=1}^n \| \theta^k - \theta^{k-1} \|^2_{L^2(\Omega)} H^{-1} \sum_{k=1}^n \Delta t^k (a[\theta^k], [\theta^k])_\gamma.
\]
(3.22)
\[
\sum_{k=1}^n \Delta t^k (aB(\xi^k - \xi^{k-1}), [\theta^k])_\gamma \leq \Delta t \sum_{k=1}^n \Delta t^k \| \partial_t \xi^k \|^2_{L^2(\Omega)} H^{-1} \sum_{k=1}^n \Delta t^k (a[\theta^k], [\theta^k])_\gamma.
\]
(3.23)
and
\[
\sum_{k=1}^n \Delta t^k (aB(u^k - u^{k-1}), [\theta^k])_\gamma \leq \Delta t \sum_{k=1}^n \Delta t^k \| \partial_t u^k \|^2_{L^2(\Omega)} H^{-1} \sum_{k=1}^n \Delta t^k (a[\theta^k], [\theta^k])_\gamma.
\]
(3.24)
From (3.6), we see that
\[
\left\| \frac{\partial u^k}{\partial n_\gamma} - B(u^k) \right\|_{L^2(\Gamma)} \leq CH^2 \| u^k \|_{W^{3,\infty}(G)}
\]
such that
\[
\sum_{k=1}^n \Delta t^k \left| \left( a \left( \frac{\partial u^k}{\partial n_\gamma} - B(u^k) \right), [\theta^k] \right)_\gamma \right| \leq CH^2 \sum_{k=1}^n \Delta t^k \| u^k \|_{W^{3,\infty}(G)} \| \theta^k \|_{L^2(\Gamma)}
\]
\[
\leq CH^2 \sum_{k=1}^n \Delta t^k \| u^k \|_{W^{3,\infty}(G)} H^{-1} \sum_{k=1}^n \Delta t^k (a[\theta^k], [\theta^k])_\gamma.
\]
Substituting these estimates into (3.21), we have
\[
\| \theta^n \|^2_{L^2(\Omega)} + \sum_{k=1}^n \left( \| \theta^k - \theta^{k-1} \|^2_{L^2(\Omega)} + \frac{1}{3} \sum_{j=1}^f (a \nabla \theta^k, \nabla \theta^k)_{\gamma} + \frac{1}{24H} (a[\theta^k], [\theta^k])_\gamma \right)
\]
\[
\leq \| \theta^0 \|^2_{L^2(\Omega)} + \frac{1}{3} \Delta t \sum_{j=1}^f (a \nabla \theta^k, \nabla \theta^k)_{\gamma} + \frac{1}{24H} (a[\theta^k], [\theta^k])_\gamma + \left\{ \sum_{k=1}^n \Delta t^k \left( \| \theta^k \|^2_{L^2(\Omega)} + \| \partial_t \xi^k \|^2_{L^2(\Omega)} + \| \xi^k \|^2_{L^2(\Omega)} \right) + \Delta t^2 + H^3 \right\}.
\]
(3.25)
If \(0 < 4a_1 \Delta t H^{-2} \leq 1\), then
\[
\| \theta^n \|^2_{L^2(\Omega)} \leq C \left\{ \sum_{k=1}^n \Delta t^k \left( \| \theta^k \|^2_{L^2(\Omega)} + \| \partial_t \xi^k \|^2_{L^2(\Omega)} + \| \xi^k \|^2_{L^2(\Omega)} \right) + \Delta t^2 + H^3 \right\}.
\]
(3.26)
Applying discrete Bellman’s inequality to (3.26) leads to (3.14). The proof of Lemma 3.6 ends.

Next, we estimate the bound of \( \xi^n \).

Lemma 3.7. There holds the a priori error estimate:
\[
\| [W - \tilde{W}] \|_{L^2(\Gamma)} \leq C \| \eta \|_{L^\infty(\Omega)}.
\]
(3.27)
Proof. We prove
\[
\|W - \tilde{W}\| \leq CH^{-\frac{3}{2}} \|\eta\|_{L^\infty(\Omega)} \tag{3.28}
\]
such that (3.27) holds. Since
\[
D(W - \tilde{W}, V) + \kappa(W - \tilde{W}, V) - (aB(W - \tilde{W}), [V])_F = D(W - u, V) + \kappa(W - u, V) - (aB(W - u), [V])_F \\
= - (aB(W - u), [V])_F, \quad \forall V \in \mathcal{M}^h.
\tag{3.29}
\]
From Lemma 3.3, we have
\[
\alpha \|W - \tilde{W}\|^2 \leq D(W - \tilde{W}, W - \tilde{W}) + \kappa(W - \tilde{W}, W - \tilde{W}) - (aB(W - \tilde{W}), [W - \tilde{W}])_F \\
= - (aB(W - u), [W - \tilde{W}])_F.
\tag{3.30}
\]
It follows from (3.1) that
\[
\left| (aB(u - W), [W - \tilde{W}])_F \right| \leq CH^{-1} \|u - W\|_{L^\infty(\Omega)} \|W - \tilde{W}\|_{L^2(\Gamma)}.
\tag{3.31}
\]
Substituting (3.31) into (3.30) leads to (3.28). □

Lemma 3.8. There hold the a priori error estimates:
\[
\|\xi\|_{L^2(\Omega)} \leq C \left\{ h^2 \|u\|_{H^2(\Omega)} + \|\eta\|_{L^\infty(\Omega)} \right\} \tag{3.32}
\]
and
\[
\|\xi_t\|_{L^2(\Omega)} \leq C \left\{ h^2 \left( \|u\|_{H^2(\Omega)} + \|u_t\|_{H^1(\Omega)} \right) + \|\eta_t\|_{L^\infty(\Omega)} \right\} \tag{3.33}
\]
Proof. Firstly, we prove
\[
\|\xi\| \leq Ch \|u\|_{H^2(\Omega)}. \tag{3.34}
\]
From the definition (3.11), we have that for each \( v \in \mathcal{M}^h \)
\[
\frac{1}{6} \|\tilde{W} - v\|^2 + \|\tilde{W} - v\|_{L^2(\Omega)}^2 \leq D(\tilde{W} - v, \tilde{W} - v) + \kappa(\tilde{W} - v, \tilde{W} - v) - (aB(\tilde{W} - v), [\tilde{W} - v])_F \\
= D(u - v, \tilde{W} - v) + \kappa(u - v, \tilde{W} - v) - (aB(u - v), [\tilde{W} - v])_F \\
\leq C \left( \|\nabla(u - v)\|_{L^2(\Omega)}^2 + \|u - v\|_{L^2(\Omega)}^2 + H \|B(u - v)\|_{L^2(\Gamma)}^2 \right)^{1/2} \left( \|\tilde{W} - v\|^2 + \|\tilde{W} - v\|_{L^2(\Omega)}^2 \right)^{1/2}
\]
such that
\[
\inf_{v \in \mathcal{M}^h} \|\tilde{W} - v\| \leq C \inf_{v \in \mathcal{M}^h} \left\{ \|u - v\|_{H^1(\Omega)} + H^{1/2} \|B(u - v)\|_{L^2(\Gamma)} \right\} \leq C \inf_{v \in \mathcal{M}^h} \left\{ \|u - v\|_{H^1(\Omega)} + H^{-1/2} \|u - v\|_{L^2(\Omega)} \right\} \leq Ch \|u\|_{H^2(\Omega)}.
\]
Hence, we get
\[
\|\xi\| \leq \inf_{v \in \mathcal{M}^h} \left\{ \|\tilde{W} - v\| + \|u - v\| \right\} \leq Ch \|u\|_{H^2(\Omega)}.
\]
This implies (3.34).
Secondly, we consider \( L^2 \)-norm error estimate. Introduce an auxiliary function \( \xi \in H^1(\Omega) \) such that
\[
- \nabla \cdot (a\nabla \xi) + \kappa \xi = \xi, \quad \text{in} \ \Omega; \quad \frac{\partial \xi}{\partial n_\Omega} = 0.
\tag{3.35}
\]
It is well known that \( \xi \in H^2(\Omega) \) and satisfies
\[
\|\xi\|_{H^2(\Omega)} \leq C \|\xi\|_{L^2(\Omega)}.
\]
By using $\zeta$, we have
\[
\|\xi\|_{L^2(\Omega)} = D(\xi + \zeta) + \kappa(\xi + \zeta) - \left( a \frac{\partial \xi}{\partial n_{\Gamma}} \right)_{\Gamma} - (aB(\xi), [\zeta])_{\Gamma} \\
= D(\xi - I^h \zeta, \xi + \kappa(\xi - I^h \zeta) + (aB(\zeta), [\zeta - I^h \zeta])_{\Gamma} - \left( a \frac{\partial \xi}{\partial n_{\Gamma}} \right)_{\Gamma}.
\] (3.36)
where $I^h$ is the interpolating operator from $H^1(\Omega)$ on to $\mathcal{M}^h$. We bound the terms on the right-hand side of (3.36). It is clear that
\[
D(\xi - I^h \zeta, \xi + \kappa(\xi - I^h \zeta) - (aB(\zeta), [\zeta - I^h \zeta])_{\Gamma} \leq C h \|\xi\|_{H^2(\Omega)} \|\xi\| \leq C h^2 \|\xi\|_{L^2(\Omega)} \|u\|_{H^2(\Omega)}.
\] (3.37)
From (3.7) and (3.27), we see that
\[
\left| \left( a \frac{\partial \xi}{\partial n_{\Gamma}} \right)_{\Gamma} \right| \leq C \|\xi\|_{H^2(\Omega)} \left( |||W - W|||_{L^2(\Gamma)} + |||W - u|||_{L^2(\Gamma)} \right) \leq C \|\xi\|_{L^2(\Omega)} \|\eta\|_{L^\infty(\Gamma)}.
\] (3.38)
From estimates above, we get the a priori error estimate (3.32). Similarly, we can get (3.33). The proof of Lemma 3.8 is complete.

Now we can prove Theorem 2.1. Noting that
\[
\|u^n - U^n\|_{L^2(\Omega)} \leq \|\xi^n\|_{L^2(\Omega)} + \|\theta^n\|_{L^2(\Omega)}
\] and applying Lemmas 3.6 and 3.8, we derive (2.13).

3.3. Proof of Theorem 2.2

To obtain an optimal error estimate of the algorithm (IDDP), we introduce another elliptic projection $\tilde{W} \in \mathcal{M}^h$ of the solution $u$ as follows:
\[
D(u - \tilde{W}, v) + \kappa(u - \tilde{W}, v) - (aB(u - \tilde{W}), [v])_{\Gamma} - (|u - \tilde{W}|, aB(v))_{\Gamma} = 0, \quad \forall \, v \in \mathcal{M}^h.
\] (3.39)
Denote error functions again by
\[
\xi = u - \tilde{W}, \quad \theta^n = U^n - \tilde{W}.
\] (3.40)
We only need to estimate $\xi$ and $\theta^n$, respectively. We firstly estimate $\theta^n$.

Lemma 3.9. There holds the a priori error estimate:
\[
\max_{0 \leq n \leq N} \|\theta^n\|_{L^2(\Omega)}^2 \leq C \sum_{n=1}^N \Delta t^n \left( \|\xi^n\|_{L^2(\Omega)}^2 + \|\partial_t \xi^n\|_{L^2(\Omega)}^2 + H \Delta t^k \|\tilde{\xi}^k\|_{L^2(\Gamma)}^2 + h^4 + \Delta t^2 + H^3 \right),
\] (3.41)
provided that
\[
\Delta t \leq \frac{H^2}{4a_1} \left( 1 - \sqrt{\frac{2}{3}} \right).
\]
Proof. It follows from (3.39) that
\[
\begin{align*}
&\left< \partial_t \tilde{W}, v \right> + D(\tilde{W}, v) + (b\tilde{W}, v) - (aB(\tilde{W}), [v])_{\Gamma} - (aB(v), [\tilde{W}])_{\Gamma} \\
&= (f^n + (\kappa + b)\xi^n + a\rho^n - a\tilde{B}, v) - \left( a \left( B(u^n) - \frac{\partial u^n}{\partial n_{\Gamma}} \right), [v] \right)_{\Gamma}, \quad \forall \, v \in \mathcal{M}^h.
\end{align*}
\] (3.42)
(2.8) and (3.42) give
\[
\begin{align*}
&\left< \partial_t \theta^n, v \right> + D(\theta^n, v) + \kappa(\theta^n, v) - (aB(\theta^n), [v])_{\Gamma} - (aB(v), [\theta^n])_{\Gamma} \\
&= \left< \partial_t \dot{\xi}^n - \rho^n - (\kappa + b)\xi^n + (\kappa - b)\theta^n, v \right> - \left( a \left( B(u^n) - B(u^n - u^{n-1}) \right), [v] \right)_{\Gamma} \\
&\quad - (aB(v), [\theta^n - \theta^{n-1}])_{\Gamma} - \left( a \left( \frac{\partial u^n}{\partial n_{\Gamma}} - B(u^n) \right), [v] \right)_{\Gamma} + (aB(\xi^n - \xi^{n-1}), [v])_{\Gamma} \\
&\quad + (aB(v), [\xi^n - \xi^{n-1}])_{\Gamma} - (aB(u^n - u^{n-1}), [v])_{\Gamma}, \quad \forall \, v \in \mathcal{M}^h.
\end{align*}
\] (3.43)
Taking \( v = \theta^n - \theta^{n-1} = \tilde{\theta}_t \theta^n \Delta t^n \) in (3.43), we have
\[
(\tilde{\theta}_t \theta^n, \tilde{\theta}_t \theta^n) \Delta t^n + D(\theta^n, \theta^n - \theta^{n-1}) + \kappa(\theta^n, \theta^n - \theta^{n-1}) - (aB(\theta^n), [\theta^n - \theta^{n-1}])_R - (aB(\theta^n - \theta^{n-1}), [\theta^n])_R \\
= (\tilde{\theta}_t \xi^n - \rho^n - (\kappa + b) \xi^n + (\kappa - b) \theta^n, \theta^n - \theta^{n-1}) - 2(aB(\theta^n - \theta^{n-1}), [\theta^n - \theta^{n-1}])_R \\
- \left( a \left( \frac{\partial u^n}{\partial n} - B(u^n) \right), [\theta^n - \theta^{n-1}]_r \right)_R + (aB(\xi^n - \xi^{n-1}), [\theta^n - \theta^{n-1}])_R \\
+ (aB(\theta^n - \theta^{n-1}), [\xi^n - \xi^{n-1}])_R - (aB(u^n - u^{n-1}), [\theta^n - \theta^{n-1}])_R. \tag{3.44}
\]
Noting that
\[
D(\theta^n, \theta^n - \theta^{n-1}) = \frac{1}{2} \left( D(\theta^n, \theta^n) - D(\theta^{n-1}, \theta^{n-1}) + D(\theta^n - \theta^{n-1}, \theta^n - \theta^{n-1}) \right)
\]
and
\[
(aB(\theta^n), [\theta^n - \theta^{n-1}])_R + (aB(\theta^n - \theta^{n-1}), [\theta^n])_R \\
= (aB(\theta^n), [\theta^n])_R - (aB(\theta^n - \theta^{n-1}), [\theta^n])_R + (aB(\theta^n - \theta^{n-1}), [\theta^n - \theta^{n-1}])_R,
\]
we have
\[
(\tilde{\theta}_t \theta^n, \tilde{\theta}_t \theta^n) \Delta t^n + \frac{1}{2} \left\{ D(\theta^n, \theta^n) + \kappa(\theta^n, \theta^n) - 2(aB(\theta^n), [\theta^n])_R \\
+ (\Delta t^n)^2 \left( D(\tilde{\theta}_t \theta^n, \tilde{\theta}_t \theta^n) + \kappa(\tilde{\theta}_t \theta^n, \tilde{\theta}_t \theta^n) - (aB(\tilde{\theta}_t \theta^n), [\tilde{\theta}_t \theta^n])_R \right) \right\} \\
= \frac{1}{2} \left( D(\theta^n, \theta^n - \theta^{n-1}) + \kappa(\theta^n, \theta^n - \theta^{n-1}) - 2(aB(\theta^n - \theta^{n-1}), [\theta^n])_R \right) \\
+ \Delta t^n \left( (\tilde{\theta}_t \xi^n - \rho^n - (\kappa + b) \xi^n + (\kappa - b) \theta^n, \tilde{\theta}_t \theta^n) - \frac{3}{2} \Delta t^n (aB(\tilde{\theta}_t \theta^n), [\tilde{\theta}_t \theta^n])_R \\
- \left( a \left( \frac{\partial u^n}{\partial n} - B(u^n) \right), [\tilde{\theta}_t \theta^n]_R \right) + \Delta t^n (aB(\tilde{\theta}_t \xi^n), [\tilde{\theta}_t \theta^n])_R \\
+ \Delta t^n (aB(\tilde{\theta}_t \theta^n), [\tilde{\theta}_t \xi^n])_R - \Delta t^n (aB(\tilde{\theta}_t u^n), [\tilde{\theta}_t \theta^n])_R \right). \tag{3.45}
\]
Summing (3.45) over \( n \), we get
\[
\sum_{k=1}^{n} (\tilde{\theta}_t \theta^k, \tilde{\theta}_t \theta^k) \Delta t^k + \frac{1}{2} \left( D(\theta^n, \theta^n) + \kappa(\theta^n, \theta^n) - 2(aB(\theta^n), [\theta^n])_R \\n+ \sum_{k=1}^{n} (\Delta t^k)^2 \left( D(\tilde{\theta}_t \theta^k, \tilde{\theta}_t \theta^k) + \kappa(\tilde{\theta}_t \theta^k, \tilde{\theta}_t \theta^k) - (aB(\tilde{\theta}_t \theta^k), [\tilde{\theta}_t \theta^k])_R \right) \right) \right\} \\
= \frac{1}{2} \left( D(\theta^n, \theta^n) + \kappa(\theta^n, \theta^n) - 2(aB(\theta^n), [\theta^n])_R \right) \\
+ \sum_{k=1}^{n} \Delta t^k \left( (\tilde{\theta}_t \xi^k - \rho^k - (\kappa + b) \xi^k + (\kappa - b) \theta^k, \tilde{\theta}_t \theta^k) - \frac{3 \Delta t^k}{2} (aB(\tilde{\theta}_t \theta^k), [\tilde{\theta}_t \theta^k])_R \\
- \left( a \left( \frac{\partial u^k}{\partial n} - B(u^k) \right), [\tilde{\theta}_t \theta^k]_R \right) + \Delta t^k (aB(\tilde{\theta}_t \xi^k), [\tilde{\theta}_t \theta^k])_R \\
+ \Delta t^k (aB(\tilde{\theta}_t \theta^k), [\tilde{\theta}_t \xi^k])_R - \Delta t^k (aB(\tilde{\theta}_t u^k), [\tilde{\theta}_t \theta^k])_R \right). \tag{3.46}
\]
It follows from Lemma 3.5 for \( \beta = \frac{5}{6} \) that
\[
(\theta^n, \theta^n) + \frac{89}{144} H^{-1} (a[\theta^n], [\theta^n])_R \leq D(\theta^n, \theta^n) + \kappa(\theta^n, \theta^n) - (aB(\theta^n), [\theta^n])_R. \tag{3.47}
\]
We estimate the terms on the right-hand side of (3.46) one by one. From Lemma 3.1, we have the following estimates:
\[
\frac{3 \Delta t^k}{2} \left| (aB(\tilde{\theta}_t \theta^k), [\tilde{\theta}_t \theta^k])_R \right| \leq 3 \Delta t^k \sqrt{aH}^{3/2} \| \tilde{\theta}_t \theta^k \|_{L^2(\Omega)} \| \tilde{\theta}_t \theta^k \|_{L^2(\Omega)} \\
\leq \frac{11}{36} H^{-1} (a[\tilde{\theta}_t \theta^k], [\tilde{\theta}_t \theta^k])_R + \frac{81}{11} (\Delta t^k)^2 a_1 H^{-2} \| \tilde{\theta}_t \theta^k \|_{L^2(\Omega)}^2,
\]
\[
(a \left( \frac{\partial u^k}{\partial n} - B(u^k) \right), [\tilde{\theta}_t^k]) \leq \frac{1}{1728} H^{-1}(a[\tilde{\alpha}_t^k], [\tilde{\theta}_t^k])_R + CH\left| \frac{\partial u^k}{\partial n} - B(u^k) \right|_{L^2(\Omega)},
\]
\[
\Delta t^k \left( aB(\tilde{\alpha}_t^k), [\tilde{\theta}_t^k] \right)_R \leq \frac{1}{1728} H^{-1}(a[\tilde{\alpha}_t^k], [\tilde{\theta}_t^k])_R + C(\Delta t^k)^2 H^{-2} \| \tilde{\eta}_t^k \|_{L^2(\Omega)},
\]
\[
\Delta t^k \left( aB(\tilde{\alpha}_t^k), [\tilde{\theta}_t^k] \right)_R \leq \epsilon \left\{ H^{-1} \| [\tilde{\theta}_t^k] \|_{L^2(\Gamma)}^2 + \sum_{j=1}^l \left( a\nabla \tilde{\alpha}_t^k, \nabla \tilde{\theta}_t^k \right)_{\Omega} \right\} + C(\Delta t^k)^2 H^{-1} \| [\tilde{\eta}_t^k] \|_{L^2(\Gamma)}^2
\]
and
\[
\Delta t^k \left( aB(\tilde{\alpha}_t^k), [\tilde{\theta}_t^k] \right)_R \leq \frac{1}{1728} H^{-1}(a[\tilde{\alpha}_t^k], [\tilde{\theta}_t^k])_R + C(\Delta t^k)^2 \| \tilde{\theta}_t^k \|_{W^{1,\infty}(\Omega)}^2.
\]

If \( 8\alpha_1 \Delta t H^{-2} \leq 1 \), then
\[
\| \theta^n \|_{L^2(\Omega)}^2 + \sum_{k=1}^n \Delta t^k \| \nabla \theta^k \|_{L^2(\Omega)}^2 \leq C \left\{ \sum_{k=1}^n \Delta t^k \left( \| \theta^k \|_{L^2(\Omega)}^2 + \sum_{i=1}^k \Delta t^i \| \theta^i \|_{L^2(\Omega)}^2 + \| \xi^k \|_{L^2(\Omega)}^2 \right) + \frac{1}{4} H \Delta t^k \| [\tilde{\eta}_t^k] \|_{L^2(\Gamma)}^2 + \| [\tilde{\theta}_t^k] \|_{L^2(\Omega)}^2 \right\}.
\]

Applying the discrete Bellman’s inequality to (3.48), we get (3.41). \( \blacksquare \)

Next, we bound \( \xi^k \) by the following lemmas.

**Lemma 3.10.** Let \( \tilde{W} \) be defined by (3.39). For each \( p \geq 2 \), there holds a priori error estimate:
\[
\|[W - \tilde{W}]\|_{L^2(\Gamma)} \leq C \left\{ H^{-\frac{1}{2}} \| \eta \|_{L^2(\Omega)} + H^{1-\frac{3}{p}} \| \nabla \eta \|_{L^p(\Omega)} \right\}.
\]

**Proof.** We prove
\[
\|[W - \tilde{W}]\|_{L^2(\Gamma)} \leq C \left\{ H^{-1} \left[ \| \eta \|_{L^2(\Omega)} + \| \nabla \eta \|_{L^2(\Omega)} \right] + H^{1-\frac{3}{p}} \| \nabla \eta \|_{L^p(\Omega)} \right\}
\]
such that (3.49) holds. Since
\[
\begin{align*}
D(W - \tilde{W}, u) + \kappa (W - \tilde{W}, W - \tilde{W}) &- (aB(W - \tilde{W}), [v])_R - ([W - \tilde{W}], aB(v))_R \\
&= D(W - u, v) + \kappa (W - u, W - u) - (aB(W - u), [v])_R - ([W - u], aB(v))_R \\
&= - (aB(W - u), [v])_R - ([W - u], aB(v))_R, \quad \forall v \in \mathcal{M}^h,
\end{align*}
\]
we have
\[
\alpha \|[W - \tilde{W}]\|_{L^2(\Omega)}^2 \leq D(W - \tilde{W}, W - \tilde{W}) + \kappa (W - \tilde{W}, W - \tilde{W}) - 2(aB(W - \tilde{W}), [W - \tilde{W}])_R \\
&= - (aB(W - u), [W - \tilde{W}])_R - ([W - u], aB(W - \tilde{W}))_R.
\]
We estimate two terms on the right-hand side of (3.51). It follows from (2.5) that
\[
||aB(W - u), [W - \tilde{W}]||_{L^2(\Gamma)} \leq CH^{-\frac{1}{2}} \| u - W \|_{L^2(\Omega)} \| [W - \tilde{W}] \|_{L^2(\Gamma)}.
\]
On the other hand, it follows from (3.4) that
\[
||[W - u]||_{L^2(\Gamma)} \leq C \left\{ H^{-1/2} \| W - u \|_{L^2(\Omega)} + H^{1-1/p} \| \nabla (W - u) \|_{L^p(\Omega)} \right\}.
\]

The proof of Lemma 3.10 is complete. \( \blacksquare \)

**Lemma 3.11.** There hold a priori error estimates:
\[
\| \xi \|_{L^2(\Omega)} \leq C \left\{ h^2 \| u \|_{H^2(\Omega)} + \| \eta \|_{L^2(\Omega)} + H^{5/4} \| \nabla \eta \|_{L^2(\Omega)} \right\}
\]
and
\[
\| \tilde{\xi} \|_{L^2(\Omega)} \leq C \left\{ h^2 \| u \|_{H^2(\Omega)} + \| \eta \|_{L^2(\Omega)} + H^{5/4} \| \nabla \eta \|_{L^2(\Omega)} \right\}.
\]
Proof. Firstly, we prove
\[ \| \xi \| \leq C h \| u \|_{H^2(\Omega)}. \] (3.55)
From the definition of (3.39), we have that for each \( v \in \mathcal{M}^h \)
\[ C_0 \| \xi \|^2 + \| \xi \|^2_{L^2(\Omega)} \leq D(\xi, \xi) + (\xi, \xi) + 2(aB(\xi), [\xi])_R \]
\[ = \inf_{v \in \mathcal{M}^h} \left\{ D(\xi, u - v) + (\xi, u - v) + (aB(\xi), [u - v])_R + ([\xi], aB(u - v))_R \right\}. \] (3.56)
Noting that
\[ \left| (aB(\xi), [u - I^h u])_R \right| \leq C h^{\frac{3}{2}} H^{-\frac{3}{2}} \| u \|_{H^2(\Gamma)} \left\{ H^{-\frac{1}{2}} \| [\xi] \|_{L^2(\Gamma)} + \| \nabla \xi \|_{L^2(\Omega)} \right\}, \] (3.57)
and
\[ (aB(u - I^h u), [\xi])_R = H^{-1}(a[u - I^h u], [\xi])_R + \left( a \int_0^1 \phi(x)(u - I^h u)(x, y)dx, [\xi] \right)_R \]
\[ \leq C_0 4 h^{\frac{2}{3}} \| [\xi] \|_{L^2(\Gamma)}^2 + C \left\{ h^2 H^{-1} \| u \|_{H^2(\Gamma)}^2 + h^2 \| u \|_{H^2(\Omega)}^2 \right\}. \] (3.58)
we obtain
\[ \| \xi \| + \| \xi \|_{L^2(\Omega)} \leq C h \| u \|_{H^2(\Omega)}. \] (3.59)
This implies (3.55).
Secondly, we consider \( L^2 \)-norm error estimate. Introduce an auxiliary function \( \zeta \in H^1(\Omega) \) such that
\[ - \nabla \cdot (a \nabla \zeta) + \zeta = \xi, \quad \text{in } \Omega; \quad \frac{\partial \zeta}{\partial n_\Omega} = 0, \quad \text{on } \partial \Omega. \] (3.60)
It is well known that \( \xi \in H^2(\Omega) \) and satisfies
\[ \| \xi \|_{H^2(\Omega)} \leq C \| \xi \|_{L^2(\Omega)}. \] (3.61)
By using \( \zeta \), we have
\[ \| \xi \|^2_{L^2(\Omega)} = D(\xi, \xi) + (\xi, \xi) + \left( \frac{\partial \zeta}{\partial n_\Gamma}, [\xi] \right)_R + (aB(\xi), [\xi])_R \]
\[ = D(\zeta - I^h \zeta, \xi) + (\zeta - I^h \zeta, \xi) + (aB(\xi), [\xi - I^h \zeta])_R + (aB(\zeta - I^h \zeta), [\xi])_R \]
\[ + \left( a \left( \frac{\partial \zeta}{\partial n_\Gamma} - B(\zeta) \right), [\xi] \right)_R. \] (3.62)
We bound the terms on the right-hand side of (3.62). It is clear that
\[ D(\zeta - I^h \zeta, \xi) + (\zeta - I^h \zeta, \xi) \leq Ch \| \xi \|_{H^2(\Omega)} \| \xi \|_{H^1(\Omega)} \] (3.63)
and
\[ (aB(\xi), [\xi - I^h \zeta])_R + (aB(\zeta - I^h \zeta), [\xi])_R \]
\[ \leq Ch \left( h^\frac{1}{2} H^{-\frac{1}{2}} \| \xi \|_{H^2(\Omega)} + \| \xi \|_{L^2(\Omega)} \right) \left( H^{-\frac{1}{2}} \| [\xi] \|_{L^2(\Gamma)} + \| \nabla \xi \|_{L^2(\Omega)} \right) \]
\[ \leq Ch \| \xi \|_{H^2(\Omega)} \| u \|_{H^2(\Omega)}. \] (3.64)
On the other hand, we have
\[ \left| \left( a \left( \frac{\partial \zeta}{\partial n_\Gamma} - B(\zeta) \right), [\xi] \right)_R \right| \leq \left| \left( a \left( \frac{\partial \zeta}{\partial n_\Gamma} - B(\zeta) \right), [u - W] \right)_R \right| + \left| \left( a \left( \frac{\partial \zeta}{\partial n_\Gamma} - B(\zeta) \right), [W - \widetilde{W}] \right)_R \right|. \] (3.65)
It follows from (3.49) that
\[ \left| \left( a \left( \frac{\partial \zeta}{\partial n_\Gamma} - B(\zeta) \right), [W - \widetilde{W}] \right)_R \right| \leq Ch \| \xi \|_{H^2(\Omega)} \| [W - \widetilde{W}] \|_{L^2(\Gamma)} \]
\[ \leq C \| \xi \|_{H^2(\Omega)} \left\{ \| \eta \|_{L^2(\Omega)} + H^{5/4} \| \nabla \eta \|_{L^4(\Omega)} \right\}. \]
We get the \textit{a priori} error estimate
\[
\|\xi\|_{L^2(\Omega)} \leq C \left[ h^2 \|u\|_{H^1(\Omega)} + \|\eta\|_{L^2(\Omega)} + H^{5/4} \|\nabla \eta\|_{L^4(\Omega)} \right].
\]
This is (3.53). Since for any \( v \in \mathcal{M}^h \)
\[
D((u - W)_t, v) + \{(u - W)_t, [v]_f + \{(u - W)_t, aB(v))_f = 0
\]
and
\[
D((u - \tilde{W})_t, v) + \{(u - \tilde{W})_t, [v]_f + \{(u - \tilde{W})_t, aB(v))_f = 0,
\]

hence we can similarly get (3.54). The proof of \textbf{Lemma 3.11} is complete. \qed

Now we can prove \textbf{Theorem 2.2}.

\textbf{Proof of Theorem 2.2.} Applying the results in \textbf{Lemma 3.11}, we have
\[
\max_{1 \leq n \leq N} \|u^n - U^n\|_{L^2(\Omega)}^2 \leq C \left\{ H^{4-\frac{4}{\beta}} \sum_{k=1}^M \Delta t^4 \left( \|\nabla \eta^k\|_{L^p(\Omega)}^2 + \|\nabla \eta^k\|_{L^p(\Omega)}^2 \right) + h^4 + \Delta t^2 + H^5 \right\}
\]
\[
\leq C \left\{ H^{4-\frac{4}{\beta}} \int_0^T \left( \|\nabla \eta\|_{L^2(\Omega)}^2 + \|\nabla \eta\|_{L^2(\Omega)}^2 \right) dt + h^4 + \Delta t^2 + H^5 \right\},
\]
which means
\[
\max_{1 \leq n \leq N} \|u^n - U^n\|_{L^2(\Omega)} \leq C \left\{ H^{2-\frac{2}{\beta}} (\|\nabla \eta\|_{L^2(0,T;L^p(\Omega))} + \|\nabla \eta\|_{L^2(0,T;L^p(\Omega))}) + h^4 + \Delta t + H^{5/2} \right\}.
\]
Taking \( p = 8/3 \), we have
\[
\max_{1 \leq n \leq N} \|u^n - U^n\|_{L^2(\Omega)} \leq C \left\{ H^{\frac{5}{2}} (\|\nabla \eta\|_{L^2(0,T;L^{8/3}(\Omega))} + \|\nabla \eta\|_{L^2(0,T;L^{8/3}(\Omega))}) + h^2 + \Delta t + H^{5/2} \right\}.
\]
The proof of \textbf{Theorem 2.2} is complete. \qed

\textbf{Acknowledgements}

We wish to express our thanks to the referees for their useful suggestions and comments.

\textbf{References}