WELLPOSEDNESS OF VARIABLE-COEFFICIENT CONSERVATIVE FRACTIONAL ELLIPTIC DIFFERENTIAL EQUATIONS

HONG WANG† AND DANPING YANG‡

Abstract. Fractional diffusion equations describe phenomena exhibiting anomalous diffusion that cannot be modeled accurately by second-order diffusion equations. Fractional differential equations raise mathematical difficulties that have not been encountered in the analysis of second-order differential equations. There are two properties of fractional differential operators that make the analysis of fractional differential equations more complicated than that for second-order differential equations. These are (i) fractional differential operators are nonlocal operators, and (ii) the adjoint of a fractional differential operator is not the negative of itself. The wellposedness of a Galerkin weak formulation to fractional elliptic differential equations with a constant diffusivity coefficient and the error analysis for corresponding finite element methods were proved previously. Many subsequent works were carried out to extend the analysis to other numerical methods. A constant diffusivity coefficient has been assumed in all these works. In this paper we present a counterexample which shows that the Galerkin weak formulation loses coercivity in the context of variable-coefficient conservative fractional elliptic differential equations. Hence, the previous results cannot be extended to variable-coefficient conservative fractional elliptic differential equations. We adopt an alternative approach to prove the existence and uniqueness of the classical solution to the variable-coefficient conservative fractional elliptic differential equation and characterize the solution in terms of the classical solutions to second-order elliptic differential equations. Furthermore, we derive a Petrov–Galerkin weak formulation to the fractional elliptic differential equation. We prove that the bilinear form of the Petrov–Galerkin weak formulation is weakly coercive and so the weak formulation has a unique weak solution and is well posed. Finally, we outline potential application of these results in the development of numerical methods for variable-coefficient conservative fractional elliptic differential equations.

Key words. classical solution, existence, Galerkin weak formulation, Petrov–Galerkin weak formulation, uniqueness, weak coercivity, weak solution, wellposedness

AMS subject classifications. 65M25, 65M60, 65Z05, 76M10, 76M25, 80A10, 80A30

DOI. 10.1137/120892295

1. Introduction. Diffusion processes occur ubiquitously in nature, science, social science, engineering, and many other disciplines. The underlying assumption for the classical Fickian diffusion process is that a particle's motion has no spatial correlation, so the variance of a particle excursion distance is finite. Then the central limit theorem concludes that the probability of finding a particle somewhere in space satisfies a normal distribution, which gives a probabilistic description of a diffusion process. In the last few decades increasingly more diffusion processes were found to be non-Fickian [2, 17, 19]. In these cases the overall motion of a particle is better described by steps that might be dependent and that can take vastly different times to perform. A particle's long walk may have a long correlation length so the process can have large deviations from the stochastic process of Brownian motion. Recent studies show that fractional diffusion equations provide an adequate and accurate...
description of transport processes that exhibit anomalous diffusion, which cannot be modeled properly by second-order diffusion equations [2, 17, 19]. Subsequently, fractional diffusion equations have been used in increasingly more applications.

Analytical methods, such as the Fourier transform method and the Laplace transform method, have been used to obtain closed-form solutions to fractional partial differential equations [19, 20]. However, as in the case of integer-order differential equations, there are only very few cases of fractional partial differential equations in which the closed-form analytical solutions are available. Therefore, numerical means have to be used in general. In the last decade different numerical methods have been developed for space-fractional partial differential equations. Liu, Anh, and Turner [12] and Meerschaert and Tadjeran [15] were the first in the development of numerical methods for fractional partial differential equations. Since then, extensive research has been carried out on the development of numerical methods for space-fractional partial differential equations [4, 6, 7, 10, 13, 14, 16, 21, 22, 23, 24, 25].

However, theoretical results on the wellposedness of fractional partial differential equations (especially with variable coefficients) are meager. The fundamental reason is that fractional differential equations raise mathematical difficulties that have not been encountered in the analysis of second-order differential equations. These are (i) fractional differential operators are nonlocal operators, and (ii) the adjoint of a fractional differential operator is not the negative of itself.

Ervin and Roop overcame these difficulties and carried out a rigorous analysis to prove the wellposedness of a Galerkin weak formulation to fractional elliptic differential equations with a constant diffusivity coefficient. In addition, they proved the optimal-order convergence rates for the corresponding finite element methods [6, 7]. Subsequently, many researchers extended the analysis to other methods such as the discontinuous Galerkin method and the spectral method [3, 4, 11]. However, all of these results require that the diffusivity coefficient be positive constant. Since variable-coefficient conservative fractional elliptic differential equations arise in many applications [2, 17], a natural question is whether and how the theoretical results in [6, 7] can be extended to these problems.

In this paper we present a counterexample to show that the bilinear form of the Galerkin weak formulation [6] is negative for a variable diffusivity coefficient (refer to Lemma 3.2)! This raises the question on the wellposedness of variable-coefficient conservative fractional elliptic differential equations as well as the corresponding Galerkin weak formulation and finite element methods! The goal of this paper is to study the wellposedness of a variable-coefficient conservative fractional elliptic differential equation and its weak formulation. It is our plan to study the corresponding finite element methods and their convergence analysis in the follow-up paper. The rest of the paper is organized as follows: In section 2 we present a one-dimensional one-sided variable-coefficient conservative fractional elliptic differential equation to be studied in this paper. In section 3 we recall the existing results for the Galerkin weak formulation with a constant diffusivity coefficient which were proved in [6]. We then present a counterexample to show that the bilinear form of the weak formulation is indefinite, which implies that the results in [6] cannot be extended to variable-coefficient problems. In section 4 we adopt a novel approach to characterize the classical solution to the variable-coefficient conservative fractional elliptic differential equation in terms of the classical solutions to second-order elliptic differential equations. This enables us to prove the wellposedness of the variable-coefficient conservative fractional elliptic differential equation in section 2. In section 5 we derive a Petrov–Galerkin weak formulation for the variable-coefficient conservative fractional elliptic differential...
equation. We prove that the bilinear form of the Petrov–Galerkin weak formulation is weakly coercive and so the weak formulation has a unique weak solution and is well posed. We again characterize the weak solution in terms of the weak solutions to second-order elliptic differential equations. In section 6 we discuss the potential impact of the theoretical results proved in this paper on the development of faithful and efficient numerical methods for variable-coefficient conservative fractional elliptic differential equations.

2. Problem formulation. In this paper we study the Dirichlet boundary-value problem of the one-sided variable-coefficient conservative fractional elliptic differential equation in one space dimension [2, 6, 12, 15]

\[ -D(K(x) D_x^{-\beta} u(x)) = f(x), \quad x \in (0, 1), \]

\[ u(0) = u_l, \quad u(1) = u_r. \]

Here \( Du(x) := u’(x) \) is the first-order differential operator, \( 2 - \beta \) with \( 0 < \beta < 1 \) represents the order of anomalous diffusion of the problem, \( K \) is the diffusivity coefficient with positive lower and upper bounds \( K_{\min} \) and \( K_{\max} \), i.e.,

\[ 0 < K_{\min} \leq K(x) \leq K_{\max} < +\infty, \quad x \in [0, 1], \]

\( f \) is the source term, and \( u_l \) and \( u_r \) are the prescribed Dirichlet boundary data. The left and right fractional integrals of order \( \beta \) are defined by [19, 20]

\[ _0D_x^{-\beta} u(x) := \frac{1}{\Gamma(\beta)} \int_0^x \frac{u(s)}{(x-s)^{1-\beta}} ds, \]

\[ _xD_1^{-\beta} u(x) := \frac{1}{\Gamma(\beta)} \int_x^1 \frac{u(s)}{(s-x)^{1-\beta}} ds, \]

where \( \Gamma(\cdot) \) is the Gamma function. The left and right fractional derivatives of Caputo form of order \( \beta \) are defined by [19]

\[ _0C D_x^\beta u(x) := _0D_x^{(1-\beta)} Du(x) = \frac{1}{\Gamma(1-\beta)} \int_0^x \frac{u’(s)}{(x-s)^\beta} ds, \]

\[ _xC D_1^\beta u(x) := -_xD_1^{(1-\beta)} Du(x) = \frac{1}{\Gamma(1-\beta)} \int_x^1 \frac{u’(s)}{(s-x)^\beta} ds. \]

The left and right fractional derivatives of Riemann–Liouville form of order \( \beta \) are defined by

\[ _0D_x^\beta u(x) := D_0D_x^{(1-\beta)} u(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_0^x \frac{u(s)}{(x-s)^\beta} ds, \]

\[ _xD_1^\beta u(x) := -xD_1^{(1-\beta)} u(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_x^1 \frac{u(s)}{(s-x)^\beta} ds. \]

Let \( u_B := u_l(1-x) + u_r x \). Then \( u_f := u - u_B \) satisfies a problem similar to (2.1) but with homogeneous Dirichlet boundary conditions and a perturbed right-hand side where an extra term \( D(K(x) _0D_x^{-\beta} Du_B) \) is added. Once this problem is solved for \( u_f \), the solution \( u \) to the original problem (2.1) is computed by \( u = u_f + u_B \). Therefore, from now on we focus on the problem with homogeneous Dirichlet
boundary conditions. For convenience of later reference, we will drop the subscript \( f \) in \( u_f \) and the perturbation of the right-hand side, and still express the problem as

\[
-D(K(x) \partial_x^{-\beta} D u) = f(x), \quad x \in (0,1),
\]

(2.6)

\( u(0) = u(1) = 0. \)

**Remark 2.1.** The term \( \partial_x^{-\beta} D u \) in (2.6) can be expressed in terms of the Caputo derivative \( \partial_x^{1-\beta} u \), which can in turn be represented in terms of the Riemann–Liouville derivative \( \partial_x^{1-\beta} u \) in the current context (refer to Lemma 4.2).

3. **Previous results for constant-coefficient problems.** We recall existing results for fractional elliptic differential equations with constant diffusivity coefficients. Then we discuss the obstacles to extend the results to variable-coefficient problems.

Ervin and Roop first carried out a rigorous analysis of the following two-sided analogue of problem (2.6) but with a constant diffusivity coefficient [6],

\[
-D(K(\theta \partial_x^{-\beta} + (1-\theta) \partial_x^{1-\beta}) D u) = f(x), \quad x \in (0,1),
\]

(3.1)

\( u(0) = u(1) = 0. \)

Here \( K \) is a positive constant. The parameter \( 0 \leq \theta \leq 1 \) indicates the relative weight of forward versus backward transition probability.

Let \( H^\mu_0(0,1) \) with \( \mu > 1/2 \) be the fractional Sobolev space of order \( \mu \) with trace 0 at the boundary \( x = 0 \) and 1. Multiplying (3.1) by any test function \( v \in H^\mu_0(0,1) \) yields the following weak formulation,

\[
\int_0^1 K(\theta \partial_x^{-\beta} u + (1-\theta) \partial_x^{1-\beta}) D u D v(x) dx = \int_0^1 f(x) v(x) dx.
\]

(3.2)

They introduced the associated bilinear form \( B : H^{1-\beta}_0(0,1) \times H^{1-\beta}_0(0,1) \to \mathbb{R} \) such that

\[
B(u,v) := \theta K(\partial_x^{-\beta} D u, D v) + (1-\theta) K(\partial_x^{1-\beta} D u, D v),
\]

(3.3)

where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing of \( H^{-\mu}(0,1) \) and \( H^{\mu}_0(0,1), \mu \geq 0 \). For a given \( f \in H^{-(1-\beta)}(0,1) \), we define the associated bounded linear functional \( l : H^{1-\beta}_0(0,1) \to \mathbb{R} \) as

\[
l(v) := \langle f, v \rangle.
\]

(3.4)

The following theorem was proved in [6].

**Theorem 3.1.** \( B(\cdot,\cdot) \) is coercive and continuous on \( H^{1-\beta}_0(0,1) \times H^{1-\beta}_0(0,1) \), i.e., there exist two positive constants \( B_{\min} \) and \( B_{\max} \) such that

\[
B(w, w) \geq B_{\min} \| w \|^2_{H^{1-\beta}_0(0,1)} \quad \forall w \in H^{1-\beta}_0(0,1),
\]

(3.5)

\[
|B(w,v)| \leq B_{\max} \| w \|_{H^{1-\beta}_0(0,1)} \| v \|_{H^{1-\beta}_0(0,1)} \quad \forall w, v \in H^{1-\beta}_0(0,1).
\]

Hence, the Galerkin weak formulation of finding \( u \in H^{1-\beta}_0(0,1) \) such that

\[
B(u,v) = l(v) \quad \forall v \in H^{1-\beta}_0(0,1)
\]

(3.6)
has a unique solution. Moreover,

\begin{equation}
\|u\|_{H^{1-\frac{\beta}{2}}(0,1)} \leq \frac{\|f\|_{H^{-\frac{1}{2}}(0,1)}}{B_{\min}}.
\end{equation}

In addition, Ervin and Roop defined a finite element scheme based on the Galerkin weak formulation (3.6) and proved the following optimal-order error estimate for the finite element approximation \( u_h \in S_h^m(0,1) \):

\begin{equation}
\|u_h - u\|_{L^2(0,1)} + h^{1-\frac{\beta}{2}} \|u_h - u\|_{H^{1-\frac{\beta}{2}}(0,1)} \leq C h^{m+1} \|u\|_{H^{m+1}(0,1)},
\end{equation}

\begin{equation*}
u \in H^{m+1}(0,1) \cap H^{1-\frac{\beta}{2}}(0,1).
\end{equation*}

Here \( S_h^m(0,1) \subset H^{1-\frac{\beta}{2}}(0,1) \) consists of continuous and piecewise polynomials of degree up to \( m \). Subsequently, these results have been extended to multidimensional problems and time-dependent problems as well as other methods such as discontinuous Galerkin methods and spectral methods. Nevertheless, all of these results require that the diffusivity coefficient \( K \) be positive constant.

Since mathematical models for anomalous diffusion processes with variable diffusivity coefficient or fractional potential equations with heterogeneous permeability occur in many applications [2, 17], a natural question arises: Can Theorem 3.1 be extended to variable-coefficient problems? More specifically, is the bilinear form \( B(\cdot, \cdot) \) corresponding to the Galerkin weak formulation (3.6) coercive on \( H^{1-\frac{\beta}{2}}_0(0,1) \times H^{1-\frac{\beta}{2}}_0(0,1) \)? Unfortunately, the answer is negative as stated in the following lemma.

**Lemma 3.2.** There exist a variable diffusivity coefficient \( K(x) \) with positive lower and upper bounds and a function \( w \in L^{1-\frac{\beta}{2}}(0,1) \) such that \( B(w, w) < 0 \). In fact, \( K \) can be chosen as piecewise constant with just two pieces.

**Proof.** We prove the lemma by construction. Let the diffusivity coefficient \( K(x) \) be defined by

\begin{equation}
K(x) := \begin{cases} K_1, & x \in (0, 1/2), \\ 1, & x \in (1/2, 1), \end{cases}
\end{equation}

We choose \( w \in H^{1-\frac{\beta}{2}}_0(0,1) \subset H^{1-\frac{\beta}{2}}(0,1) \) to be a continuous and piecewise-linear function

\begin{equation}
w(x) := \begin{cases} 2x, & x \in (0, 1/2], \\ 2(1 - x), & x \in [1/2, 1). \end{cases}
\end{equation}

Direct calculation gives

\begin{equation}
_{0}D_x^{1-\beta} w(x) = \begin{cases} 2x^\beta / \Gamma(\beta + 1), & x \in (0, 1/2), \\ 2(x^\beta - 2(x - 1)^\beta) / \Gamma(\beta + 1), & x \in (1/2, 1). \end{cases}
\end{equation}

Then we have

\begin{equation}
B(w, w) = \frac{2^{1-\beta}}{\Gamma(\beta + 2)} (K_1 - (2^\beta + 1)).
\end{equation}
Since \(0 < \log_2 3 - 1 < 1\), choose \(\log_2 3 - 1 < \beta < 1\) so that \(2^{2\beta+1} - 3 > 0\). Then we select \(K_l > 0\) sufficiently small such that \(K_l - (2^{2\beta+1} - 3) < 0\). For such \(K\) and \(w\), we have \(B(w, w) < 0\).

**Remark 3.1.** We can modify the diffusivity \(K(x)\) so that \(K_l\) and 1 can be connected smoothly and monotonely increasing with a transitional zone of width \(\varepsilon\). As \(\varepsilon \to 0^+\), we can find a smooth and positive diffusivity \(K(x)\) and the function \(w\) defined in (3.10) such that the corresponding \(B(w, w) < 0\).

4. Existence, uniqueness, and characterization of the classical solution to variable-coefficient problem (2.6). The counterexample in Lemma 3.2 shows that the Galerkin weak formulation (3.6) is not the right framework for variable-coefficient problem (2.6) and corresponding numerical methods. In this section we adopt a different approach to prove the existence and uniqueness of the classical solution to the variable-coefficient conservative fractional elliptic differential equation (2.6). We also characterize the solution by means of the solutions to second-order elliptic differential equations.

Let \(C[0, 1]\) be the space of continuous functions on the interval \([0, 1]\). For any positive integer \(m\) and \(0 < \mu \leq 1\), let

\[
C^{0, \mu}[0, 1] := \left\{ v \in C[0, 1] : \max_{x, y \in [0, 1], x \neq y} \frac{|v(x) - v(y)|}{|x - y|^{\mu}} < \infty \right\},
\]

\[
C^m[0, 1] := \left\{ v \in C[0, 1] : D^k v \in C[0, 1], \quad 0 \leq k \leq m \right\},
\]

\[
C^{m, \mu}[0, 1] := \left\{ v \in C^m[0, 1] : D^m v \in C^{\mu}[0, 1] \right\}
\]
equipped with the (semi)norms defined by

\[
|v|_{C^{0, \mu}[0, 1]} := \max_{x, y \in [0, 1], x \neq y} \frac{|v(x) - v(y)|}{|x - y|^{\mu}},
\]

\[
\|v\|_{C^{0, \mu}[0, 1]} := \|v\|_{C[0, 1]} + |v|_{C^{0, \mu}[0, 1]},
\]

\[
\|v\|_{C^m[0, 1]} := \max_{0 \leq k \leq m} \|D^k v\|_{C[0, 1]},
\]

\[
\|v\|_{C^{m, \mu}[0, 1]} := \|v\|_{C^m[0, 1]} + \|D^m v\|_{C^{\mu}[0, 1]}.
\]

In addition, we introduce fractional-order continuous function spaces and Hölder function spaces for \(0 < \beta, \mu \leq 1\),

\[
C^{1-\beta}[0, 1] := \left\{ v \in C[0, 1] : 0D_x^{1-\beta} v \in C[0, 1] \right\},
\]

\[
C^{m-\beta}[0, 1] := \left\{ v \in C^{m-1}[0, 1] : 0D_x^{m-\beta} v \in C^{\mu}[0, 1] \right\},
\]

\[
C^{1-\beta, \mu}[0, 1] := \left\{ v \in C^{1-\beta}[0, 1] : 0D_x^{1-\beta} v \in C^{\mu}[0, 1] \right\},
\]

\[
C^{m-\beta, \mu}[0, 1] := \left\{ v \in C^{m-\beta}[0, 1] : 0D_x^{m-\beta} v \in C^{\mu}[0, 1] \right\}
\]
equipped with the norms

\[
\|v\|_{C^{m-\beta}[0, 1]} := \|v\|_{C^{m-\beta}[0, 1]} + \|D_x^{m-\beta} v\|_{C[0, 1]},
\]

\[
\|v\|_{C^{m-\beta, \mu}[0, 1]} := \|v\|_{C^{m-\beta}[0, 1]} + \|D_x^{m-\beta} v\|_{C^{\mu}[0, 1]}.
\]
4.1. Auxiliary lemmas. We recall and prove several lemmas that will be used in the proof of the main results on the classical solution to problem (2.6) in section 4.2.

The following lemma was proved in [19, 20].

Lemma 4.1. The left and right Riemann–Liouville fractional integral operators are adjoints in the $L^2$ sense, i.e., for all $\mu > 0$,

\[
(0D_x^{-\beta} w, v)_{L^2(0,1)} = (w, xD_1^{-\beta} v)_{L^2(0,1)} \quad \forall w, v \in L^2(0,1).
\]

The left and right Riemann–Liouville fractional integral operators follow the properties of a semigroup, i.e., for any $w \in L^p(0,1)$ with $p \geq 1$,

\[
0D_x^{-\mu} 0D_x^{-\sigma} w(x) = 0D_x^{-\mu-\sigma} w(x) \quad \forall x \in [0,1], \forall \mu, \sigma > 0,
\]

\[
xD_1^{-\mu} xD_1^{-\sigma} w(x) = xD_1^{-\mu-\sigma} w(x) \quad \forall x \in [0,1], \forall \mu, \sigma > 0.
\]

Lemma 4.2. Let $0 < \mu \leq 1$. Then for any function $w \in C^1[0,1]$ with $w(0) = 0$,

\[
\int_0^x D^\mu_x w(x) = 0D^\mu_x w(x).
\]

For any function $w \in C^1[0,1]$ with $w(1) = 0$,

\[
\int_0^1 D^\mu_x w(x) = xD^\mu_x w(x).
\]

Proof. We only prove (4.7), since (4.8) can be proved similarly. For $\mu = 1$, both fractional derivatives reduce to the first-order classical derivative. So (4.7) holds trivially. We now consider the case $0 < \mu < 1$. In this case,

\[
0D^\mu_x w(x) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_0^x \frac{w(s)}{(x-s)^\mu} ds
\]

\[
= \frac{1}{\Gamma(2-\mu)} \frac{d}{dx} \left[ -\frac{x^{1-\mu} w(s)}{\Gamma(1-\mu)} \bigg|_{s=0}^{s=x} + \int_0^x (x-s)^{1-\mu} w'(s) ds \right]
\]

\[
= \frac{1}{\Gamma(1-\mu)} \int_0^x (x-s)^{-\mu} w'(s) ds = \int_0^1 D^\mu_x w(x).
\]

This concludes the proof of the lemma. \[\square\]

Remark 4.1. Under the condition of the lemma, we do not need to distinguish the fractional derivative of Caputo form from that of Riemann–Liouville form.

Lemma 4.3. Let $0 < \mu < 1$. Then for any $w \in C^1[0,1]$ with $w(0) = 0$, we have

\[
0D^\mu_x w \bigg|_{x=0} = 0
\]

and

\[
0D^\mu_x 0D_x^{-\mu} w = 0D_x^{-\mu} 0D^\mu_x w = w.
\]

Similarly, for any $w \in C^1[0,1]$ with $w(1) = 0$, we have

\[
xD^\mu_x w \bigg|_{x=1} = 0
\]

and

\[
xD_1^{-\mu} xD_1^\mu w = xD_1^{-\mu} xD_1^\mu w = w.
\]
That is, under the condition of the lemma, \( D_0^\mu \) and \( D_1^\mu \) are both the left and the right inverse of \( D_x^{-\mu} \) and \( D_1^{-\mu} \), respectively, and vice versa.

Proof. We conclude from (4.7) that

\[
\tag{4.14}
0D_x^\mu w = \frac{1}{\Gamma(1-\mu)} \int_0^x (x-s)^{-\mu}w'(s)ds \to 0 \quad \text{as } x \to 0^+.
\]

As for (4.11) we use (2.5) and the semigroup property (4.6) to obtain

\[
\tag{4.15}
0D_0^{-\mu}D_x^{-\mu}w = D_0D_x^{-(1-\mu)}D_x^{-\mu}w = D_0D_x^{-1}w = w.
\]

On the other hand, we use (4.7) to deduce

\[
\tag{4.16}
0D_x^{-\mu}D_0^\mu w = 0D_x^{-\mu}D_x^{-\mu}w = 0D_x^{-\mu}D_x^{-(1-\mu)}Dw
\]

The proof of (4.12) and (4.13) is similar and is omitted here.

Lemma 4.4. Let \( 0 < \mu \leq 1 \). The following mapping properties hold:

\begin{enumerate}
\item \( D_x^{-\mu} : C[0,1] \to C^0,\mu[0,1] \) is a bounded linear operator.
\item \( D_x^\mu : C^1[0,1] \to C^{0,1-\mu}[0,1] \) is a bounded linear operator.
\end{enumerate}

Proof. For \( \mu = 1 \) the fractional integrals and derivatives reduced to the integrals and derivatives in the classical sense. So the properties hold trivially. We need only to consider the case \( 0 < \mu < 1 \). To prove (i) we see directly from definition (2.3) that \( 0D_x^{-\mu}w \) is continuous with respect to \( x \). So we need only to check that \( 0D_x^{-\mu}w \) is Hölder continuous of order \( \mu \) with respect to \( x \). By symmetry we assume that \( 0 \leq y < x \leq 1 \).

\[
\tag{4.17}
\left| 0D_x^{-\mu}w(x) - 0D_x^{-\mu}w(y) \right|
= \frac{1}{\Gamma(\mu)} \left| \int_0^x \frac{w(s)}{(x-s)^{1-\mu}}ds - \int_0^y \frac{w(s)}{(y-s)^{1-\mu}}ds \right|
= \frac{1}{\Gamma(\mu)} \left| \int_y^x \frac{w(s)}{(x-s)^{1-\mu}}ds + \int_y^x \frac{w(s)}{(x-s)^{1-\mu}} - \frac{w(s)}{(y-s)^{1-\mu}}ds \right|
\leq \frac{||w||_{C[0,1]}}{\Gamma(\mu)} \left( \frac{|x-y|^{\mu}}{\mu} + \int_y^x \frac{1}{(y-s)^{1-\mu}} - \frac{1}{(x-s)^{1-\mu}} ds \right)
\leq \frac{||w||_{C[0,1]}}{\Gamma(\mu + 1)} \left( (x-y)^{\mu} + y^{\mu} + (x-y)^{\mu} - x^{\mu} \right)
\leq \frac{2||w||_{C[0,1]}}{\Gamma(\mu + 1)} |x-y|^{\mu}.
\]

Here in the first “\( \leq \)” sign we have used the fact that \( (x-s)^{\mu} > (y-s)^{\mu} \). In the last “\( \leq \)” sign we have used the fact that \( 0 \leq y < x \leq 1 \) so \( y^{\mu} < x^{\mu} \).

We can prove (ii) in a similar manner. We see directly from definition (2.4) and (4.7) that \( 0D_x^\mu w \) is continuous with respect to \( x \). So we need only to check that \( 0D_x^\mu w \) is Hölder continuous of order \( 1-\mu \) with respect to \( x \). Let \( 0 \leq y < x \leq 1 \). Then we
have

\[
|D_x^\mu w(x) - D_y^\mu w(y)| = \left| \frac{1}{\Gamma(1-\mu)} \int_0^x \frac{w'(s)}{(x-s)^\mu} ds - \frac{1}{\Gamma(1-\mu)} \int_0^y \frac{w'(s)}{(y-s)^\mu} ds \right|
\]

\[
= \frac{1}{\Gamma(1-\mu)} \left| \int_0^x \frac{w'(s)}{(x-s)^\mu} ds + \int_0^y \frac{w'(s)}{(y-s)^\mu} ds \right|
\]

\[
\leq \frac{\|w\|_{C^1[0,1]} \left( |x-y|^{1-\mu} + \int_0^y \left( \frac{1}{(y-s)^\mu} - \frac{1}{(x-s)^\mu} \right) ds \right)}{\Gamma(2-\mu)}
\]

\[
\leq \frac{2\|w\|_{C^1[0,1]} |x-y|^{1-\mu}}{\Gamma(2-\mu)}.
\]  

We thus conclude the proof of the lemma.

\[\Box\]

**4.2. The main results on the classical solution.** In this subsection we prove the existence, uniqueness, and characterization of the classical solution to the variable-coefficient conservative fractional elliptic differential equation (2.6).

**Theorem 4.5.** Let \(0 < \alpha \leq 1\) and \(0 < \beta < 1\). Assume that \(K \in C^{1,\alpha}[0,1]\) and \(f \in C^{\alpha}[0,1]\). Then \(u \in C^{2-\beta,\alpha}[0,1]\) is the unique classical solution to the Dirichlet boundary-value problem of the variable-coefficient conservative fractional elliptic differential equation (2.6) if and only if it can be expressed as

\[
u = D_x^{\beta} w_f - D_1^{\beta} w_f (D_1^{\beta} w_b)^{-1} D_x^{\beta} w_b,
\]

where \(w_f\) and \(w_b\) are the classical solutions to the Dirichlet boundary-value problems of the following second-order elliptic differential equations

\[
-D(K(x) Dw_f) = f, \quad x \in (0,1),
\]

\[
w_f(0) = w_f(1) = 0,
\]

and

\[
-D(K(x) Dw_b) = 0, \quad x \in (0,1),
\]

\[
w_b(0) = 0, \quad w_b(1) = 1.
\]

We begin by proving the following lemma.

**Lemma 4.6.** The classical solution \(w_b\) to the boundary-value problem (4.21) satisfies

\[
0D_x^2 w_b \bigg|_{x=1} > 0.
\]

**Proof.** Direct calculation shows that

\[
w_b = \left( \int_0^1 \frac{1}{K(s)} ds \right)^{-1} \int_0^x \frac{1}{K(s)} ds.
\]
We apply (4.14) to \( w_b \) to get
\[
(4.24) \quad _0D_x^2 w_b = \left[ \int_0^1 \frac{1}{K(s)} ds \right]^{-1} \frac{1}{\Gamma(1 - \beta)} \int_0^x \frac{1}{K(s)(x - s)\beta} ds.
\]
In particular,
\[
(4.25) \quad _0D_x^\beta w_b \bigg|_{x=1} = \left[ \int_0^1 \frac{1}{K(s)} ds \right]^{-1} \frac{1}{\Gamma(1 - \beta)} \int_0^1 \frac{1}{K(s)(1 - s)\beta} ds > 0.
\]
This concludes the proof.

We are now in a position to prove Theorem 4.5.

Proof. We first check the regularity of \( u \) given by (4.19). By the theory of second-order differential equations [8, 9], we conclude that the classical solutions \( w_f \) and \( w_b \) to the boundary-value problems (4.20) and (4.21) belong to \( C^{2,\alpha}[0,1] \). It is clear from (4.19) that \( u \in C^1[0,1] \). We apply \( _0D_x^{2-\beta} \) to both sides of (4.19) and utilize (4.11) to deduce that
\[
(4.26) \quad _0D_x^{2-\beta} u = D^2 w_f - (_0D_x^1 w_f)(_0D_x^1 w_b)^{-1}D^2 w_b \in C^{\alpha}[0,1].
\]
Thus, \( u \in C^{2,\alpha}[0,1] \).

Next we check that the function \( u \) given by the expression (4.19) is a solution to problem (2.6). We apply (4.14) to get \( u(0) = 0 \). On the other hand, direct calculation shows that
\[
(4.27) \quad u(1) = _0D_x^1 w_f(1) - _0D_x^1 w_f(1)(_0D_x^1 w_b)^{-1} _0D_x^1 w_b(1) = 0.
\]
We apply (4.11) to (4.19) to obtain
\[
(4.28) \quad _0D_x^{-\beta} u = w_f - _0D_x^1 w_f(_0D_x^1 w_b)^{-1} w_b.
\]
With this equation we arrive at
\[
(4.29) \quad -D(K(x) _0D_x^{1-\beta} u) = -D(K(x) Dw_f) + _0D_x^1 w_f(_0D_x^1 w_b)^{-1} D(K(x) Dw_b) = f(x) + 0 = f(x), \quad x \in (0,1).
\]
This shows that the function \( u \) given by the expression (4.19) is indeed a classical solution to problem (2.6).

On the other hand, let \( u \) be a classical solution to problem (2.6). We introduce an auxiliary function \( w \) defined by
\[
(4.30) \quad w := _0D_x^{-\beta} u.
\]
Then the function \( w \) is the classical solution to the following Dirichlet boundary-value problem of a second-order elliptic differential equation:
\[
(4.31) \quad -D(K(x) Dw) = f, \quad x \in (0,1),
\]
\[
 w(0) = 0, \quad w(1) = _0D_x^{-\beta} u.
\]
Let the functions \( w_f \) and \( w_b \) be the classical solutions to the Dirichlet boundary-value problems of the second-order elliptic differential equations (4.20) and (4.21),
respectively. Then the solution $w$ can be expressed as a linear combination of $w_f$ and $w_b$,

\begin{equation}
    w = w_f + Cw_b.
\end{equation}

We apply (4.11) to (4.30) and then use (4.32) to obtain

\begin{equation}
    u = 0D_x^\beta w_f + C 0D_x^\beta w_b.
\end{equation}

We enforce the homogeneous boundary condition in problem (2.6) for $u$ to get

\begin{equation}
    0D_1^\beta w_f(1) + C 0D_1^\beta w_b(1) = 0.
\end{equation}

Lemma 4.6 shows that $D_1^\beta w_b(1) \neq 0$. Hence,

\begin{equation}
    C = -\left(0D_1^\beta w_b(1)\right)^{-1} 0D_1^\beta w_f(1).
\end{equation}

We substitute the constant $C$ in (4.35) into the expression (4.33) to conclude that the resulting solution $u$ given by (4.33) and (4.35) actually coincides with the expression (4.19). In other words, any solution $u$ to problem (2.6) can be expressed in the form of (4.19).

Finally, we prove the uniqueness of the solution to problem (2.6). We need only to prove that the following homogeneous analogue to problem (2.6),

\begin{equation}
    -D(K(x) 0D_x^{1-\beta} u) = 0, \quad x \in (0, 1),
\end{equation}

\begin{equation}
    u(0) = u(1) = 0,
\end{equation}

has only the trivial solution. We use the change of variable (4.30) to express (4.36) in terms of $w$ as a boundary-value problem of a second-order elliptic differential equation

\begin{equation}
    -D(K(x)Dw) = 0, \quad x \in (0, 1),
\end{equation}

\begin{equation}
    w(0) = 0, \quad w(1) = 0D_1^{1-\beta} u.
\end{equation}

It is clear that

\begin{equation}
    w = (0D_1^{-\beta} u) w_b
\end{equation}

with $w_b$ being the solution to problem (4.21). We apply (4.11) to (4.30) to obtain

\begin{equation}
    u = 0D_x^\beta w = (0D_1^{-\beta} u) 0D_x^\beta w_b.
\end{equation}

Since $u(1) = 0$ and by Lemma 4.6 $0D_x^\beta w_b|_{x=1} \neq 0$, we have $0D_1^{-\beta} u = 0$. Namely, problem (4.37) has a homogeneous boundary condition at both $x = 0$ and $x = 1$. Therefore, $w \equiv 0$ in $[0, 1]$. Then (4.39) shows that $u \equiv 0$ in $[0, 1]$. That is, the homogeneous problem (4.36) has only the trivial solution. Therefore, the solution to problem (2.6) is unique.

5. Wellposedness of a Petrov–Galerkin weak formulation. The counterexample in Lemma 3.2 shows that for a variable diffusivity coefficient the bilinear form $B$ of Galerkin weak formulation (3.6) is not coercive in $H_0^{1-\frac{\beta}{2}}(0, 1) \times H_0^{1-\frac{\beta}{2}}(0, 1)$ or any product space $H \times H$. This raises questions on the existence and uniqueness.
of the weak solution to Galerkin weak formulation (3.6) in the context of variable-coefficient problem (2.6). In this section we derive a Petrov–Galerkin weak formulation and prove that the bilinear form $A$ of the Petrov–Galerkin weak formulation is weakly coercive on the product space $H_0^{1-\beta}(0,1) \times H_0^1(0,1)$ so that the Petrov–Galerkin weak formulation has a unique weak solution in the fractional Sobolev space $H_0^{1-\beta}(0,1)$ and is well posed. We also characterize the weak solution in terms of the weak solutions to some second-order elliptic differential equations.

5.1. Auxiliary lemmas. We prove lemmas that will be used in the proof of the main results on the weak solution to the Petrov–Galerkin weak formulation to problem (2.6) in section 5.2.

Let $C_0^\infty(0,1)$ be the space of infinitely many times differentiable functions that vanish outside a compact subset of the interval $(0,1)$. Further, define the fractional-order seminorm

\begin{equation}
|w|_{J^\mu(0,1)} := \|0D_x^\mu w\|_{L^2(0,1)}
\end{equation}

and norm

\begin{equation}
\|w\|_{J^\mu(0,1)} := (|w|^2 + \|w\|_{J^\mu(0,1)}^2)^{1/2},
\end{equation}

and let $J^\mu_0(0,1)$ denote the space that is the closure of $C_0^\infty(0,1)$ with respect to the norm $\|w\|_{J^\mu(0,1)}$.

Lemmas 5.1–5.3 were proved in [6].

**Lemma 5.1.** Let $\mu > 0$. Then the space $J_0^\mu(0,1)$ and the fractional Sobolev space $H_0^\mu(0,1)$ are equal. Moreover, if $\mu \neq m - 1/2$ for any positive integer $m$, then the space $J_0^\mu(0,1)$ and space $H_0^\mu(0,1)$ have equivalent seminorms and norms. That is, there exist positive constants $C_{\min} = C_{\min}(\mu)$ and $C_{\max} = C_{\max}(\mu)$ such that

\begin{align}
C_{\min}|w|_{H^\mu(0,1)} \leq |w|_{J^\mu(0,1)} & \leq C_{\max}|w|_{H^\mu(0,1)}, \\
C_{\min}\|w\|_{H^\mu(0,1)} \leq \|w\|_{J^\mu(0,1)} & \leq C_{\max}\|w\|_{H^\mu(0,1)}.
\end{align}

**Lemma 5.2** (fractional Poincaré inequality). Let $\mu > 0$. For any $w \in H_0^\mu(0,1)$, we have

\begin{equation}
\|w\|_{L^2(0,1)} \leq C_{\min}\|w\|_{H^\mu(0,1)}.
\end{equation}

**Lemma 5.3.** Let $0 < \mu \leq 1$. The following mapping properties hold:

(i) $0D_x^{-\mu} : L^2(0,1) \to J^\mu(0,1)$ is a bounded linear operator.

(ii) $0D_x^\mu : J^\mu(0,1) \to L^2(0,1)$ is a bounded linear operator.

We now prove the following lemma.

**Lemma 5.4.** Let $0 \leq \beta < 1/2$. Then $0D_x^\beta : H^1[0,1] \cap \{w(0) = 0\} \to C^{0,\frac{1}{2}-\beta}[0,1] \cap \{w(0) = 0\}$ is a bounded linear operator. Furthermore, (4.10) and (4.11) still hold.

**Proof.** We first prove $0D_x^\beta w \in C^{0,\frac{1}{2}-\beta}[0,1]$ for any $w \in C^1[0,1]$ with $w(0) = 0$. By symmetry, let $0 \leq y < x \leq 1$. We use (2.5) and (4.7) to get

\begin{align}
0D_x^\beta w(x) - 0D_x^\beta w(y) &= \frac{1}{\Gamma(1-\beta)} \left[ \int_0^x w'(s) (x-s)^{-\beta} ds - \int_0^y w'(s) (y-s)^{-\beta} ds \right] \\
&= \frac{1}{\Gamma(1-\beta)} \left[ \int_y^x w'(s) (x-s)^{-\beta} ds + \int_0^y w'(s) \left( \frac{1}{(x-s)^\beta} - \frac{1}{(y-s)^\beta} \right) ds \right].
\end{align}
We apply the Cauchy–Schwarz inequality to bound the first term on the right-hand side of (5.1) by
\[
\frac{1}{\Gamma(1-\beta)} \left| \int_x^y \frac{w'(s)}{(x-s)^\beta} ds \right| \leq \left\| w \right\|_{H^\beta(1,0)} \left( \int_x^y \frac{1}{(x-s)^{2\beta}} ds \right)^{\frac{1}{2}} \\
\leq C \| w \|_{H^\beta(1,0)} |x-y|^{\frac{1}{2} - \beta}.
\]

We introduce a change of variables \( s = y - (x-y)\theta \) in the second term on the right-hand side of (5.1) to bound this term by
\[
\frac{1}{\Gamma(1-\beta)} \left| \int_0^y w'(s) \left( \frac{1}{(x-s)^\beta} - \frac{1}{(y-s)^\beta} \right) ds \right| \\
\leq \frac{\| w \|_{H^\beta(1,0)}}{\Gamma(1-\beta)} \left| x-y \right|^{\frac{1}{2} - \beta} \left( \int_0^\infty \left( \frac{1}{\theta^\beta} - \frac{1}{(1+\theta)^\beta} \right)^2 d\theta \right)^{\frac{1}{2}} \\
\leq \frac{\| w \|_{H^\beta(1,0)}}{\Gamma(1-\beta)} \left| x-y \right|^{\frac{1}{2} - \beta} \left( \int_0^\infty \frac{1}{\theta^\beta} - \frac{1}{(1+\theta)^\beta} \right)^{\frac{1}{2}}.
\]

It is clear that the improper integral on the right-hand side of (5.7) converges near \( \theta = 0 \). To ensure its convergence at \( \theta = \infty \) we just note that for \( \theta \gg 1 \)
\[
\frac{1}{\theta^\beta} - \frac{1}{(1+\theta)^\beta} = \frac{(1+\theta)^\beta - \theta^\beta}{\theta^\beta (1+\theta)^\beta} = \frac{\left( \frac{1}{\theta} + 1 \right)^\beta - 1}{\theta^\beta \left( \frac{1}{\theta} + 1 \right)^\beta} \\
= \frac{\beta + O \left( \frac{1}{\theta} \right)}{\theta^{1+\beta} \left( \frac{1}{\theta} + 1 \right)^\beta}.
\]

Hence, the improper integral on the right-hand side of (5.7) also converges at \( \theta = \infty \). We combine (5.5)–(5.7) to conclude that
\[
\left| 0 D^2_x w \right|_{C^{\frac{1}{2} - \beta}, [0,1]} \leq C \| w \|_{H^\beta(1,0)}
\]
for any \( w \in C^1[0,1] \) with \( w(0) = 0 \). In addition, we also have
\[
\left| 0 D^2_x w(x) \right| = \frac{1}{\Gamma(1-\beta)} \left| \int_0^x w'(s) \frac{1}{(x-s)^\beta} ds \right| \\
\leq \frac{\| w \|_{H^\beta(1,0)}}{\Gamma(1-\beta)} \left( \int_0^x \frac{1}{(x-s)^{2\beta}} ds \right)^{\frac{1}{2}} \\
\leq C x^{\frac{1}{2} - \beta} \| w \|_{H^\beta(1,0)}.
\]

We thus obtain
\[
\left| 0 D^2_x w \right|_{C^{\frac{1}{2} - \beta}, [0,1]} \leq C \| w \|_{H^\beta(1,0)}
\]
for any $w \in C^1[0, 1]$ with $w(0) = 0$. Since the space $C^1[0, 1] \cap \{w(0) = 0\}$ is dense in the space $H^1(0, 1) \cap \{w(0) = 0\}$, a standard density argument shows that (5.9) holds for any $w \in H^1(0, 1) \cap \{w(0) = 0\}$. That is, we have proved the embedding property of the fractional differential operator $\partial^{\beta}_x$ from the space $H^1[0, 1] \cap \{w(0) = 0\}$ to the space $C^{0, \frac{1}{2} - \beta}[0, 1] \cap \{w(0) = 0\}$.

We take the limit of (5.10) as $x \to 0^+$ to conclude that $\partial^{\beta}_x w(x) = 0$ at $x = 0$, i.e., (4.10) holds. To prove (4.11) let $w$ be any fixed function in $H^1(0, 1)$ with $w(0) = 0$. Note that (4.13) holds for any $\phi \in C_0^\infty(0, 1)$. We then apply (2.5), (4.13), and the adjoint property (4.5) to conclude that

$$
(w, \phi)_{L^2(0,1)} = (w, \partial^{\beta}_x (\partial^{1-\beta}_x \phi))_{L^2(0,1)} = (\partial^{\beta}_x w, \partial^{1-\beta}_x \phi)_{L^2(0,1)}
$$

(5.12)

Since $C_0^\infty(0, 1)$ is dense in $L^2(0,1)$, we conclude from (5.12) that $\partial^{\beta}_x \partial^{1-\beta}_x w = w$.

On the other hand, it follows from Lemma 5.4 that $\partial^{\beta}_x w \in C[0, 1]$ for any $w \in H^1(0, 1)$ with $w(0) = 0$. Thus, $\partial^{1-\beta}_x w \in C^1[0, 1]$. Further, we notice that $\partial^{1-\beta}_x \phi$ vanishes at $x = 1$. Hence, we apply (4.8) and the adjoint property (4.5) to obtain

$$
(w, \phi)_{L^2(0,1)} = (w, \partial^{1-\beta}_x (\partial^{\beta}_x \phi))_{L^2(0,1)} = (\partial^{\beta}_x w, \partial^{1-\beta}_x \phi)_{L^2(0,1)}
$$

(5.13)

Here we have used the fact that $\partial^{1-\beta}_x w$ vanishes at $x = 0$ and $\partial^{\beta}_x \phi$ vanishes at $x = 1$. Thus, we have proved that $\partial^{\beta}_x \partial^{1-\beta}_x w = w$ for any $w \in H^1(0, 1)$ with $w(0) = 0$. This concludes the proof of the lemma.

**5.2. The main results on the weak solution.** In this subsection we prove the existence, uniqueness, and characterization of the weak solution to the Petrov–Galerkin weak formulation to the variable-coefficient conservative fractional elliptic equation (2.6).

**Theorem 5.5.** Assume that $0 \leq \beta < 1/2$ and that $K \in L^\infty(0, 1)$ satisfies the condition (2.2). Then there exists a constant $\gamma = \gamma(\beta) > 0$ such that the bilinear form defined by

$$
A(w, v) := \int_0^1 K(x) \partial^{1-\beta}_x w(x) Dv(x) dx
$$

is weakly coercive in the sense that

$$
\inf_{w \in H^1_0(0,1)} \sup_{v \in H^1(0,1)} \frac{A(w, v)}{\|w\|_{H^1(0,1)} \|v\|_{H^1(0,1)}} \geq \gamma
$$

(5.14)

(5.15)
and

\begin{equation}
\sup_{w \in H^{-\beta}_{0,1}(0,1)} A(w, v) > 0 \quad \forall v \in H^1_{0,1}(0,1) \setminus \{0\}.
\end{equation}

We apply Theorem 5.5 and the Babuška–Lax–Milgram theorem \cite{1, 18} to arrive at the following corollary which shows the wellposedness of the Petrov–Galerkin weak formulation.

**Corollary 5.6.** Assume that $0 \leq \beta < 1/2$ and that $K \in L^\infty(0,1)$ satisfies the condition (2.2) and $f \in H^{-1}(0,1)$. Then the Petrov–Galerkin weak formulation of finding $u \in H^{1-\beta}_{0,1}(0,1)$ such that

\begin{equation}
A(u, v) = l(v) \quad \forall v \in H^1_{0,1}(0,1),
\end{equation}

with the linear functional $l(v)$ being defined in (3.4), has a unique weak solution $u \in H^{1-\beta}_{0,1}(0,1)$. Furthermore,

\begin{equation}
\|u\|_{H^{1-\beta}(0,1)} \leq \frac{K_{\max}}{\gamma} \|f\|_{H^{-1}(0,1)}.
\end{equation}

We are now in a position to prove Theorem 5.5.

**Proof.** We first prove (5.15). For each $w \in H^{1-\beta}_{0,1}(0,1)$, we conclude from Lemmas 5.1 and 5.3 that $\partial_x^{1-\beta} w \in L^2(0,1)$. Thus, the bilinear form $A(w, \phi)$ defined in (5.14) induces a bounded linear functional on the space $H^1_{0,1}(0,1)$. We define the following weak formulation that corresponds to a second-order elliptic problem: find $v \in H^1_{0,1}(0,1)$ such that

\begin{equation}
(KDv, D\phi)_{L^2(0,1)} = A(w, \phi) \quad \forall \phi \in H^1_{0,1}(0,1).
\end{equation}

The Lax–Milgram theorem ensures that problem (5.19) has a unique solution $v \in H^1_{0,1}(0,1)$. We substitute the expression of $A(w, \phi)$ into the right-hand side of (5.19) to obtain the following relation

\begin{equation}
(KD(v - \partial_x^{1-\beta} w), D\phi)_{L^2(0,1)} = 0 \quad \forall \phi \in H^1_{0,1}(0,1).
\end{equation}

In addition, $v - \partial_x^{1-\beta} w = 0$ at $x = 0$ and $v - \partial_x^{1-\beta} w = -\partial_x^{1-\beta} w$ at $x = 1$. Thus, we conclude that

\begin{equation}
v - \partial_x^{1-\beta} w = -\partial_x^{1-\beta} w_b.
\end{equation}

Here $w_b$ is the weak solution to the following weak formulation: seek $w_b \in H^1(0,1)$ with $w_b(0) = 0$ and $w_b(1) = 1$ such that

\begin{equation}
(KDw_b, Dw)_{L^2(0,1)} = 0 \quad \forall v \in H^1_{0,1}(0,1).
\end{equation}

It is clear that (5.22) is the weak formulation to the second-order problem (4.21).

We apply $D^\beta_x$ to both sides of the equation and use Lemma 5.4 to arrive at the following expression for $w$:

\begin{equation}
w = \partial_x^{\beta} v + (\partial_x^{1-\beta} w) \partial_x^{\beta} w_b.
\end{equation}

We note that $w(1) = 0$ and $\partial_x^{1-\beta} w_b > 0$ (refer to (4.22)). Then we enforce (5.23) at $x = 1$ to obtain

\begin{equation}
\partial_x^{1-\beta} w = -\partial_x^{\beta} v (\partial_x^{1-\beta} w_b)^{-1}.
\end{equation}
We substitute (5.24) into (5.21) to get
\begin{equation}
(5.25) \quad v - aD_x^{-\beta} w = \left( aD_1^{-\beta} v \right) \left( aD_1^{-\beta} w_b \right)^{-1} w_b.
\end{equation}
We differentiate (5.25) to obtain the following expression for \( aD_x^{-\beta} w \):
\begin{equation}
(5.26) \quad aD_x^{-\beta} w = Dv - \left( aD_1^{-\beta} v \right) \left( aD_1^{-\beta} w_b \right)^{-1} Dw_b.
\end{equation}
We then use the Poincaré inequality to bound \( aD_x^{-\beta} w \) by
\begin{equation}
(5.27) \quad \| w\|_{H^{1-\beta}(0,1)} \leq C (\| Dv\|_{L^2(0,1)} + \| aD_1^{-\beta} v \| \left( aD_1^{-\beta} w_b \right)^{-1} \| Dw_b\|_{L^2(0,1)}).
\end{equation}
Since \( aD_x^{-\beta} w = cD_x^{-\beta} v \) for any \( v \in H_0^1(0,1) \) by Lemma 4.2, we have
\begin{equation}
(5.28) \quad \| aD_1^{-\beta} v \| = \| aD_1^{-\beta} v \| = \left| \int_0^1 \frac{v'(s)}{(1-s)^{\beta}} \, ds \right| \leq C \| Dv\|_{L^2(0,1)}.
\end{equation}
The combination of (5.27) with (5.28) yields the following upper bound for \( w \):
\begin{equation}
(5.29) \quad \| w\|_{H^{1-\beta}(0,1)} \leq C \| Dv\|_{L^2(0,1)}.
\end{equation}
Consequently, we obtain the following lower bound for \( A(w,v) \):
\begin{equation}
(5.30) \quad A(w,v) = (KDv, Dw)_{L^2(0,1)} \geq K_{\min} \| Dv\|_{L^2(0,1)}^2 \geq \frac{K_{\min}}{C} \| Dv\|_{L^2(0,1)}^2 \| w\|_{H^{1-\beta}(0,1)}.
\end{equation}
This leads to the following inequality
\begin{equation}
(5.31) \quad \sup_{v \in H_0^1(0,1) \setminus \{0\}} \frac{A(w,v)}{\| v\|_{H_0^1(0,1)}} \geq \frac{K_{\min}}{C} \| w\|_{H^{1-\beta}(0,1)} \quad \forall \, w \in H_0^{1-\beta}(0,1).
\end{equation}
Thus, we prove (5.15) with \( \gamma := K_{\min}/C \).

Next, we prove (5.16). For each \( v \in H_0^1(0,1) \setminus \{0\} \), we define \( w \) by
\begin{equation}
(5.32) \quad w := aD_x^{-\beta} v - (aD_1^{-\beta} v) (aD_1^{-\beta} w_b)^{-1} (aD_x^{-\beta} w_b).
\end{equation}
It is clear that \( w \in H_0^{1-\beta}(0,1) \). Furthermore, we have
\begin{equation}
(5.33) \quad aD_x^{-\beta} w = Dw - (aD_1^{-\beta} v) (aD_1^{-\beta} w_b)^{-1} Dw_b.
\end{equation}
Therefore, we arrive at
\begin{equation}
(5.34) \quad A(w,v) = (KDv, Dw)_{L^2(0,1)} - (aD_1^{-\beta} v) (aD_1^{-\beta} w_b)^{-1} (KDw_b, Dw)_{L^2(0,1)}
\geq K_{\min} \| Dv\|_{L^2(0,1)}^2 > 0.
\end{equation}
We have thus proved (5.16). This concludes the proof of the theorem. \( \Box \)

**Theorem 5.7.** Assume that the conditions in Corollary 5.6 hold. Then \( u \in H_0^{1-\beta}(0,1) \) is the unique weak solution to the Petrov–Galerkin weak formulation (5.17) if and only if it can be expressed as
\begin{equation}
(5.35) \quad u = aD_x^{-\beta} w_f - (aD_1^{-\beta} w_f) (aD_1^{-\beta} w_b)^{-1} aD_x^{-\beta} w_b.
\end{equation}

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
Here \( w_b \) is the weak solution to the weak formulation by (5.22) and \( w_f \) is the weak solution to the following weak formulation: seek \( w_f \in H^1_0(0,1) \) such that

\[
(K Dw_f, Dv)_{L^2(0,1)} = l(v) \quad \forall \ v \in H^1_0(0,1).
\]

Here \( l(v) \) is defined in Corollary 5.6.

**Remark 5.1.** It is clear that problem (5.36) is the weak formulation to problem (4.20).

**Proof.** Since the existence and uniqueness of the weak solution to the Petrov–Galerkin weak formulation (5.17) has already been proved in Theorem 5.5 and Corollary 5.6, we only need to check that the function \( u \) defined by (5.35) is a solution to (5.17).

Standard theory concludes that the weak formulations (5.22) and (5.36) have unique solutions \( w_b \) and \( w_f \), respectively. Then the function \( u \) given by the expression (5.35) is uniquely determined. We apply \( D^{1-\beta}_x \) to (5.35) to obtain

\[
_{0}D^{1-\beta}_x u = Dw_f - (_{0}D^{\beta}_1 w_f)(_{0}D^{\beta}_1 w_b)^{-1} Dw_b \in L^2(0,1),
\]

namely, \( u \in H^{1-\beta}(0,1) \). It is clear that \( u \) satisfies the homogeneous boundary conditions. Finally, we see that

\[
A(u, w) = \langle K_{0}D^{1-\beta}_x u, Dw \rangle_{L^2(0,1)}
\]

\[
= (KDw_f, Dv) - (_{0}D^{\beta}_1 w_f)(_{0}D^{\beta}_1 w_b)^{-1} (KDw_b, Dv)_{L^2(0,1)}
\]

\[
= (KDw_f, Dv)_{L^2(0,1)} + 0 = l(v) \quad \forall \ v \in H^1_0(0,1).
\]

Thus \( u \) defined by (5.35) is indeed a solution to (5.17). This concludes the proof of the theorem. □

**6. Impact on the development of numerical methods.** The theoretical results proved in sections 4–5 not only assert the existence, uniqueness, and well-posedness of the (classical or weak) solutions to variable-coefficient fractional elliptic differential equations, but also provide a constructive approach to characterize these solutions. These characterizations in turn have potential impact on the development of faithful and efficient numerical methods for variable-coefficient conservative fractional elliptic differential equations.

Because of the nonlocal nature of fractional differential operators, numerical methods for fractional elliptic differential equations typically generate dense or even full coefficient matrices [5, 6, 7, 15]. Consequently, the numerical methods for space-fractional differential equations were traditionally solved by Gaussian elimination, which requires computational work of \( O(N^3) \) and memory of \( O(N^2) \) for a discretization with \( N \) spatial grid points. The significant computational work and memory requirement of the numerical methods for fractional differential equations impose a serious challenge for the numerical simulation of these problems in multiple space dimensions. Seeking faithful and efficient numerical methods for fractional differential equations has been one of the major goals in the research on fractional differential equations. Based on theoretical results proved in sections 4–5 we outline the development of two types of numerical methods for the efficient numerical solution of the variable-coefficient conservative fractional elliptic differential equation (2.6) (and thus (2.1)).
6.1. An indirect numerical method. The characterization formula (4.19) in Theorem 4.5 provides an alternative approach for the efficient solution of the boundary-value problem of the variable-coefficient fractional elliptic differential equation (2.6) (and thus (2.1)). In this approach we would use standard numerical methods, such as piecewise-linear finite element methods or finite volume methods, or centered finite difference methods, to solve the boundary-value problems of second-order elliptic differential equations (4.20) and (4.21) for the numerical approximations \( w_{f,h} \) and \( w_{b,h} \) to \( w_f \) and \( w_b \), respectively, where \( h := 1/N \) refers to the mesh size. Next, we define an approximation \( u_h \) to the true solution \( u \) to the fractional differential equation (2.6) as follows:

\[
\begin{align*}
(6.1) \quad u_h := 0D_x^\beta w_{f,h} - 0D_i^\beta w_{f,h}(0D_i^\beta w_{b,h})^{-1}0D_x^\beta w_{b,h}.
\end{align*}
\]

A composite quadrature with a mesh size \( h \) can be used to evaluate the integrals on the right-hand side. We subtract (4.19) from (6.1) and decompose the resulting equation as follows:

\[
\begin{align*}
(6.2) \quad u_h - u = 0D_x^\beta(w_{f,h} - w_f) - 0D_i^\beta(w_{f,h} - w_f)(0D_i^\beta w_{b,h})^{-1}0D_x^\beta w_{b,h} \\
&- 0D_i^\beta w_f((0D_i^\beta w_{b,h})^{-1} - (0D_i^\beta w_{b})^{-1})0D_x^\beta w_{b,h} \\
&- 0D_i^\beta w_f(0D_i^\beta w_{b})^{-1}0D_x^\beta(w_{b,h} - w_b).
\end{align*}
\]

Thus, the \( L^2 \) error \( ||u_h - u||_{L^2(0,1)} \) is expected to be of suboptimal order in the case that a piecewise-linear finite element method is used:

\[
(6.3) \quad ||u_h - u||_{L^2(0,1)} = O(h^{2-\beta}).
\]

Note that the numerical solution of the second-order elliptic differential equations (4.20) and (4.21) requires \( O(N) \) of computational work and memory storage. In addition, the evaluation of (6.1) also requires \( O(N) \) of computational work and storage. In summary, the indirect method proposed in this subsection reduces the computational work for the numerical solution of the variable-coefficient fractional elliptic differential equation (2.6) from \( O(N^3) \) to \( O(N) \) and the memory requirement from \( O(N^2) \) to \( O(N) \), but at the cost of a reduced convergence rate of \( O(h^{2-\beta}) \). We will present detailed development and analysis in the follow-up paper.

6.2. A Petrov–Galerkin finite element method. Theorem 5.5 and Corollary 5.6 assert that the Petrov–Galerkin weak formulation (5.17) is weakly coercive and so has a unique solution and is well posed. This motivates the development of a Petrov–Galerkin finite element method. One crucial issue in the development of a Petrov–Galerkin finite element method is a clever construction of a pair of finite-dimensional subspaces in the product space \( H_0^{1-\beta}(0,1) \times H_0^\beta(0,1) \) such that (5.15) and (5.16) hold on the finite-dimensional subspaces. So far, there is not any clue at all in the construction of such a pair of finite-dimensional subspaces. This remains the fundamental issue in the development of a Petrov–Galerkin finite element method in the future.

Another issue is the computational cost and memory requirement of the Petrov–Galerkin finite element method to be developed. Because of the nonlocal nature of fractional differential operators, the resulting Petrov–Galerkin method is expected to yield a dense or full stiffness matrix. This again requires computational work of \( O(N^3) \) and memory of \( O(N^2) \). The authors previously developed fast solution techniques for
time-dependent nonconservative space-fractional diffusion equations in the context of finite difference methods [22, 23, 24]. These methods reduce the computational work of the traditional finite difference methods from $O(N^3)$ to $O(N \log^2 N)$ per time step and the memory requirement from $O(N^2)$ to $O(N)$. Numerical experiments show the utility of the fast solution techniques. However, these solution techniques do not apply to the nonconventional Petrov–Galerkin finite element method proposed here. Hence, the development of a faithful and efficient solution technique for the proposed Petrov–Galerkin finite element method remains to be studied in the future.

Acknowledgments. The authors would like to express their sincere thanks to the editor and referees for their very helpful comments and suggestions, which greatly improved the quality of this paper.

REFERENCES


